Week 5
Lines, Planes

Dot Product

Definition.

The **dot product** of two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ is:

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v} = \sum_{i=1}^{n} v_i w_i.$$

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Notice that:

$$(\lambda \vec{a}) \cdot \vec{b} = \lambda(\vec{a} \cdot \vec{b}), \quad \lambda \in \mathbb{R}.$$

$$(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}.$$

$$\vec{v} \cdot \vec{v} = \sum_{i=1}^{n} v_i^2 = |\vec{v}|^2$$

If θ is the angle ($0 \leq \theta \leq \pi$) between two nonzero vectors $\vec{v}$ and $\vec{w}$, then:

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}| \cos\theta,$$

or equivalently,

$$\theta = \arccos\left(\frac{\vec{v}}{|\vec{v}|} \cdot \frac{\vec{w}}{|\vec{w}|}\right)$$

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If $\vec{v}$ and $\vec{w}$ are orthogonal/perpendicular to each other, then $\theta = \pi/2$; hence:

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}| \cos(\pi/2) = 0.$$

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Conversely, if $\vec{v}$, $\vec{w}$ are nonzero vectors and $\vec{v} \cdot \vec{w} = 0$, then the two vectors are orthogonal to each other.

If nonzero $\vec{v}, \vec{w}$ are parallel to each other, then:

$$\vec{v} \cdot \vec{w} = \begin{cases} |\vec{v}|\,|\vec{w}| & \text{if they point in the same direction;} \\ -|\vec{v}|\,|\vec{w}| & \text{if they point in opposite directions.} \end{cases}$$

Projection

Given two vectors $\vec{v}, \vec{w}$, where $\vec{w} \neq \vec{0}$, we can always express $\vec{v}$ as the sum of a vector $\mathrm{Proj}_{\vec{w}}\vec{v}$ which is parallel to $\vec{w}$, and a vector $\vec{v}_\perp$ which is orthogonal to $\vec{w}$:

$$\vec{v} = \mathrm{Proj}_{\vec{w}}\vec{v} + \vec{v}_\perp. \tag{$*$}$$

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To find $\mathrm{Proj}_{\vec{w}}\vec{v}$, we note that:

$$\mathrm{Proj}_{\vec{w}}\vec{v} = \lambda\vec{w}$$

for some $\lambda \in \mathbb{R}$, since $\mathrm{Proj}_{\vec{w}}\vec{v}$ and $\vec{w}$ are parallel.

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Taking the dot product of both sides of the equation $(*)$ with $\vec{w}$, we have:

$$\vec{v} \cdot \vec{w} = (\mathrm{Proj}_{\vec{w}}\vec{v} + \vec{v}_\perp) \cdot \vec{w} = \underbrace{\lambda\vec{w} \cdot \vec{w}}_{=\lambda|w|^2} + \underbrace{\vec{v}_\perp \cdot \vec{w}}_{=0}.$$

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Hence,

$$\lambda = \frac{1}{|\vec{w}|^2}\vec{v} \cdot \vec{w},$$

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so:

$$\begin{aligned} \mathrm{Proj}_{\vec{w}}\vec{v} &= \lambda\vec{w} = \left(\frac{1}{|\vec{w}|^2}\vec{v} \cdot \vec{w}\right)\vec{w} = \left(\vec{v} \cdot \frac{\vec{w}}{|\vec{w}|}\right)\frac{\vec{w}}{|\vec{w}|}, \\ \vec{v}_\perp &= \vec{v} - \mathrm{Proj}_{\vec{w}}\vec{v}. \end{aligned}$$

Note that $\frac{\vec{w}}{|\vec{w}|}$ is the unit vector associated with $\vec{w}$.

Parameterization of a line in $\mathbb{R}^n$.

Let O be the origin of $\mathbb{R}^n$. Let L be a line in $\mathbb{R}^n$ which passes through a given point $P_0 \in \mathbb{R}^n$, and is parallel to a vector $\vec{v} \in \mathbb{R}^n$. Each point P on L satisfies:

$$\overrightarrow{P_0P} = t\vec{v},$$

for some $t \in \mathbb{R}$.

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On the other hand, we have:

$$\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0}.$$

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Hence,

$$\overrightarrow{OP} = \overrightarrow{OP_0} + t\vec{v}.$$

The line L is therefore described by the vector-valued function:

$$\boxed{\vec{l}\,(t) = \overrightarrow{OP_0} + t\vec{v}, \quad t \in \mathbb{R}.}$$

Distance Between a Point and a Line

Given any point $Q \in \mathbb{R}^3$, The (minimal) distance d between the point Q and the line L is:

$$\boxed{d = \left|\overrightarrow{P_0Q} - \mathrm{Proj}_{\vec{v}}\overrightarrow{P_0Q}\right|.}$$

Planes in $\mathbb{R}^3$

Two non-parallel vectors $\vec{v}$, $\vec{w}$ in $\mathbb{R}^3$ determine a plane $\mathcal{P}$ in $\mathbb{R}^3$ containing the origin. The plane $\mathcal{P}$ consists of all points $P \in \mathbb{R}^3$ such that $\overrightarrow{OP}$ lies in the linear span of $\vec{v}$ and $\vec{w}$:

$$\overrightarrow{OP} = s\vec{v} + t\vec{w}, \quad s, t \in \mathbb{R}.$$

A plane in $\mathbb{R}^3$ containing a fixed point P_0 is the set of points $P \in \mathbb{R}^3$ such that:

$$\overrightarrow{P_0P} \in \mathrm{Span}\,\{\vec{v}, \vec{w}\} = \{s\vec{v} + t\vec{w} \mid s, t \in \mathbb{R}\},$$

where $\vec{v}$ and $\vec{w}$ are fixed non-parallel vectors.

We focus first on a plane $\mathcal{P}$ which contains the origin. In this case, $\mathcal{P}$ consists of points (x, y, z) such that:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + t\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

for some $s, t \in \mathbb{R}$.

Theorem.

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There exists a vector $\langle a, b, c\rangle \neq \vec{0}$, such that $\mathcal{P}$ consists of all points (x, y, z) which satisfy:

$$ax + by + cz = 0.$$

Conversely, given $\langle a, b, c\rangle \neq \vec{0}$, the set of points (x, y, z) which satisfy:

$$ax + by + cz = 0$$

form a plane in $\mathbb{R}^3$. That is, there are non-parallel vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$, such that (x, y, z) satisfies $ax + by + cz = 0$ if and only if:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s\vec{v} + t\vec{w},$$

for some $s, t \in \mathbb{R}$.

In general, a plane $\mathcal{P}$ (not necessarily containing the origin) is described by an equation of the form:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0, \qquad (*)$$

where $P_0 = (x_0, y_0, z_0)$ is a point which lies on $\mathcal{P}$. Note that for all $P \in \mathcal{P}$, we have $\overrightarrow{P_0P} \cdot \langle a, b, c\rangle = 0$. In other words, $\overrightarrow{P_0P}$ is perpendicular to $\langle a, b, c\rangle$. We call $\langle a, b, c\rangle$ a **normal** vector to the plane. Expanding and regrouping the terms in the equation $(*)$, a plane in $\mathbb{R}^3$ corresponds to the set of solutions (x, y, z) to an equation of the form:

$$ax + by + cz = d.$$