Week 4
Deteminants
Cramer's Rule

Properties of the Determinant

Let A be an $n \times n$ matrix.

- $$\det A = \det A^\top,$$

  where $A^\top$ is the transpose of A, defined by $A^\top_{ij} = A_{ji}$. This follows from the fact that $\det A$ may be computed from the cofactor expansion along any row or column.

- If A is an upper or lower triangular matrix, then $\det A$ is equal to the product of its diagonal entries:

  $$\det \begin{pmatrix} a_{11} & * & * & * \\ 0 & a_{22} & * & * \\ & & \ddots & \\ 0 & 0 & & * \\ 0 & 0 & 0 & a_{nn} \end{pmatrix} = a_{11}a_{22}\cdots a_{nn}$$

  In particular, the determinant of the identity matrix is equal to one.

- If one row or one column of A consists entirely of zeroes, then $\det A = 0$.

  $$\det \begin{pmatrix} 1 & 2 & 3 & -5 \\ 0 & 0 & 0 & 0 \\ 3 & 4 & -7 & 9 \\ 0 & 3 & 1 & 8 \end{pmatrix} = 0$$

  $$\det \begin{pmatrix} 1 & 2 & 0 & -5 \\ 6 & -7 & 0 & 3 \\ 3 & 4 & 0 & 9 \\ 0 & 3 & 0 & 8 \end{pmatrix} = 0$$

- If a matrix B is obtained from a square matrix A by switching two rows, then $\det B = -\det A$.

- If one row (column) of A is equal to a scalar multiple of another row (column), then $\det A = 0$.

- The determinant of an elementary matrix is nonzero.

- If E is an $n \times n$ elementary matrix, then $\det(EA) = (\det E)(\det A)$.

Theorem.

A is invertible if and only if $\det A \neq 0$.

Proof.

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By a previous result, A is invertible if and only if it is row equivalent to I.

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Suppose A is invertible, then there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that:

$$E_k \cdots E_2 E_1 A = I.$$

We have:

$$(\det E_k) \cdots (\det E_2)(\det E_1)(\det A) = \det I = 1$$

Hence, $\det A \neq 0$.

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If A is not invertible, then it is row equivalent to a matrix with a row consisting entirely of zeroes. Hence, there exist elementary matrices $E_1, E_2, \ldots E_k$, such that:

$$0 = \det(E_k \ldots E_2 E_1 A) = (\det E_k) \cdots (\det E_2)(\det E_1)(\det A).$$

Since the determinants of elementary matrices are nonzero, we conclude that $\det A = 0$.

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Remark.

Suppose a matrix A is invertible, then there are elementary matrices $E_1, \ldots, E_k$ such that:

$$E_k \cdots E_1 A = I,$$

or equivalently:

$$A = E_1^{-1} \cdots E_k^{-1}.$$

Since the inverse of an elementary matrix is an elementary matrix, we conclude that a matrix A is invertible if and only if it is a product of elementary matrices.

Theorem.

Let A, B be $n \times n$ matrices. Then,

$$\det(AB) = (\det A)(\det B).$$

Proof.

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Suppose B is non-invertible. Then, $\det(B) = 0$. Moreover, there exists a nonzero $\vec{v} \in \mathbb{R}^n$ such that $B\vec{v} = \vec{0}$. We have:

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$$(AB)\vec{v} = A(B\vec{v}) = A\vec{0} = \vec{0}.$$

In other words, the matrix equation $(AB)\vec{x} = \vec{0}$ has a nonzero solution, which implies that AB is non-invertible. Hence, $0 = \det(AB) = \det(A)\det(B)$.

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Suppose B is invertible but A is not. Then, $\det(A) = 0$, and there exists $\vec{v} \neq 0$ such that $A\vec{v} = \vec{0}$. Let $\vec{w} = B^{-1}\vec{v} \neq 0$ (note that B^{-1} is invertible).

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We have:

$$(AB)\vec{w} = (AB)B^{-1}\vec{v} = A(BB^{-1})\vec{v} = A\vec{v} = \vec{0}.$$

So, $(AB)\vec{x} = \vec{0}$ has a nonzero solution. By the same argument as before, we conclude that in this case $\det(AB) = \det(A)\det(B)$.

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Now, suppose A and B are both invertible. There are elementary matrices $E_1, \ldots, E_k$ such that $A = E_1 E_2 \cdots E_k$. Hence:

$$\det(AB) = \det(E_1 \cdots E_k B) = \det(E_1) \cdots \det(E_k) \det(B) = \det(A) \det(B).$$

This follows from the fact that $\det(EC) = \det(E) \det(C)$ for any $n \times n$ elementary matrix E and $n \times n$ matrix C.

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Note that $\det(AB) = \det(A) \det(B) = \det(BA)$, even if $AB \neq BA$.

Corollary.

If A is invertible, then:

$$\det(A^{-1}) = \frac{1}{\det A}.$$

Adjoint of a Matrix

The **adjoint** of an $n \times n$ matrix A is the $n \times n$ matrix $\text{adj}A = ((\text{adj } A)_{ij})$ defined by:

$$(\text{adj } A)_{ij} = (-1)^{i+j} |M_{ji}|.$$

$$\text{adj } A = \begin{pmatrix} (-1)^{1+1} |M_{11}| & (-1)^{1+2} |M_{21}| & \cdots & (-1)^{1+n} |M_{n1}| \\ \vdots & \cdots & \cdots & \vdots \\ (-1)^{n+1} |M_{1n}| & \cdots & \cdots & (-1)^{n+n} |M_{nn}| \end{pmatrix}$$

Example.

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$$\text{adj} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Observe that:

$$\begin{aligned} (A(\text{adj } A))_{ij} &= \sum_{k=1}^{n} A_{ik} (\text{adj } A)_{kj} \\ &= \sum_{k=1}^{n} a_{ik} (-1)^{k+j} |M_{jk}| = \begin{cases} \det A & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \end{aligned}$$

And similarly for $(\text{adj } A)A$.

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Hence,

$$A(\text{adj } A) = (\text{adj } A)A = (\det A)I_n = \begin{pmatrix} \det A & 0 & \cdots & \cdots & 0 \\ 0 & \det A & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \det A \end{pmatrix}$$

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If A is invertible, then $\det A \neq 0$, and we have:

$$\boxed{A^{-1} = \frac{1}{\det A}\text{adj } A}$$

Cramer's Rule

Suppose an $n \times n$ matrix A is invertible. To solve an equation of the form:

$$A\vec{x} = \vec{b},$$

we multiply both sides of the equation by $A^{-1} = \dfrac{1}{\det A}\text{adj } A$ from the left, obtaining:

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$$\vec{x} = \frac{1}{\det A}(\text{adj } A)\vec{b}.$$

The i-th entry of the vector $(\text{adj } A)\vec{b} \in \mathbb{R}^n$ is given by:

$$((\text{adj } A)\vec{b})_i = \sum_{k=1}^{n}(\text{adj } A)_{ik}b_k = \sum_{k=1}^{n}(-1)^{i+k}|M_{ki}|\, b_k,$$

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which is the cofactor expansion along the i-th column of the matrix A_i obtained from A by replacing the i-th column of A by $\vec{b}$.

$$A_i = \left(\begin{array}{cccc|c|ccc} a_{11} & a_{12} & \cdots & a_{1(i-1)} & b_1 & a_{1(i+1)} & \cdots & a_{1n} \\ a_{21} & a_{22} & \vdots & a_{2(i-1)} & b_2 & a_{2(i+1)} & \vdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(i-1)} & b_n & a_{n(i+1)} & \cdots & a_{nn} \end{array}\right)$$

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Hence, the i-th entry of the vector $\vec{x} = \dfrac{1}{\det A}(\text{adj } A)\vec{b}$ is:

$$x_i = \frac{\det A_i}{\det A}.$$

This is known as **Cramer's Rule**.

Example.

Use Cramer's Rule to solve the following linear system:

$$\begin{aligned} -4x_1 + 7x_2 + 7x_3 &= -8 \\ x_1 + 6x_2 + 3x_3 &= 5 \\ 6x_1 + 8x_2 - 4x_3 &= 24 \end{aligned}$$

Solution.

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This corresponds to the matrix equation:

$$\underbrace{\begin{pmatrix} -4 & 7 & 7 \\ 1 & 6 & 3 \\ 6 & 8 & -4 \end{pmatrix}}_{A} \vec{x} = \underbrace{\begin{pmatrix} -8 \\ 5 \\ 24 \end{pmatrix}}_{\vec{b}}$$

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We have:

$$|A_1| = \left| \begin{pmatrix} -8 & 7 & 7 \\ 5 & 6 & 3 \\ 24 & 8 & -4 \end{pmatrix} \right| = 300,$$

$$|A2| = \left| \begin{pmatrix} -4 & -8 & 7 \\ 1 & 5 & 3 \\ 6 & 24 & -4 \end{pmatrix} \right| = 150,$$

$$|A_3| = \left| \begin{pmatrix} -4 & 7 & -8 \\ 1 & 6 & 5 \\ 6 & 8 & 24 \end{pmatrix} \right| = -150.$$

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Hence,

$$\begin{aligned} x_1 &= \frac{|A_1|}{|A|} = \frac{300}{150} = 2 \\ x_2 &= \frac{|A_2|}{|A|} = \frac{150}{150} = 1 \\ x_3 &= \frac{|A_3|}{|A|} = \frac{-150}{150} = -1 \end{aligned}$$

Geometry of Vectors

Alternative notation for a vector in $\mathbb{R}^n$:

$$\langle v_1, v_2, \ldots, v_n \rangle = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

The **norm** (or **length**, **magnitude**) of a vector $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle \in \mathbb{R}^n$ is:

$$|\vec{v}| = \sqrt{\sum_{i=1}^{n} v_i^2}.$$

Note that:

$$|\vec{v}| = 0 \Leftrightarrow \vec{v} = \vec{0}.$$

For all $\lambda \in \mathbb{R}$,

$$\begin{aligned} |\lambda \vec{v}| &= \sqrt{\sum_{i=1}^{n} (\lambda v_i)^2} \\ &= \sqrt{\lambda^2 \sum_{i=1}^{n} v_i^2} \\ &= \sqrt{\lambda^2} \sqrt{\sum_{i=1}^{n} v_i^2} \\ &= |\lambda| \, |\vec{v}| \end{aligned}$$

Unit Vector

A vector of length 1 is called a **unit vector**.

For $\vec{v} \neq \vec{0}$, the vector $\frac{1}{|\vec{v}|}\vec{v}$ has length:

$$\left| \frac{1}{|\vec{v}|}\vec{v} \right| = \frac{1}{|\vec{v}|} |\vec{v}| = 1.$$

Definition.

We call $\frac{1}{|\vec{v}|}\vec{v}$ the **unit vector associated with** $\vec{v}$.

Intuitively speaking, the unit vector associated with $\vec{v}$ captures the direction of $\vec{v}$, and ignores its length.

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Every nonzero vector $\vec{v}$ has the form:

$$\vec{v} = \lambda\vec{u}, \quad \lambda > 0,$$

where $\vec{u} = \frac{1}{|\vec{v}|}\vec{v}$ is the unit vector associated with $\vec{v}$, and $\lambda = |\vec{v}|$ is the length of $\vec{v}$.

Dot Product

Definition.

The **dot product** of two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ is:

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v} = \sum_{i=1}^{n} v_i w_i.$$

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Notice that:

$$(\lambda\vec{a}) \cdot \vec{b} = \lambda(\vec{a} \cdot \vec{b}), \quad \lambda \in \mathbb{R}.$$

$$(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}.$$

$$\vec{v} \cdot \vec{v} = \sum_{i=1}^{n} v_i^2 = |\vec{v}|^2$$

If θ is the angle $(0 \leq \theta \leq \pi)$ between two nonzero vectors $\vec{v}$ and $\vec{w}$, then:

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}|\cos\theta,$$

or equivalently,

$$\theta = \arccos\left(\frac{\vec{v}}{|\vec{v}|} \cdot \frac{\vec{w}}{|\vec{w}|}\right)$$
