Week 11
The Gradient
Taylor's Theorem

The Gradient Vector

Definition.

Let F be a function in n variables $x_1, x_2, \ldots, x_n$. The gradient of F at $P = (a_1, a_2, \ldots, a_n)$ is the vector:

$$\langle F_{x_1}(P), F_{x_2}(P), \ldots, F_{x_n}(P) \rangle \in \mathbb{R}^n.$$

Here,

$$F_{x_i}(P) = \frac{\partial F}{\partial x_i}\Big|_{(x_1, x_2, \ldots, x_n) = (a_1, a_2, \ldots, a_n)}$$

Theorem. Let $F(x_1, x_2, \ldots, x_n)$ be a function in n variables, P a point on the level set:

$$F(x_1, x_2, \ldots, x_n) = c$$

If the gradient vector $\nabla F(P) = \langle F_{x_1}(P), F_{x_2}(P), \ldots, F_{x_n}(P) \rangle$ of F at P is nonzero, then $\nabla F(P)$ is perpendicular to the level set $F(x_1, x_2, \ldots, x_n) = c$, in the sense that it is perpendicular to the tangent vector at P to every smooth curve on $F(x_1, x_2, \ldots, x_n) = c$ which passes through P.
In other words:

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Claim.

If I is an open interval in $\mathbb{R}$, and a differentiable vector-valued function $\gamma : I \to \mathbb{R}^n$ satisfies:

$$\begin{aligned} F(\gamma(t)) &= c \quad \text{(i.e. The curve lies on the level set.)} \\ \gamma(t_0) &= P, \quad t_0 \in I, \quad \text{(i.e. The curve passes through the point } P \text{ when } t = t_0) \end{aligned}$$

then:

$$\nabla F(P) \cdot \gamma'(t_0) = 0.$$

Proof.

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Suppose $\gamma(t) = \langle \gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t) \rangle$, where γ_i is a differentiable real-valued function in one variable. Applying $\frac{d}{dt}$ to both sides of $F(\gamma(t)) = c$, we have:

$$\frac{d}{dt} F(\gamma(t)) = \frac{d}{dt} c$$
$$\underbrace{\nabla F(\gamma(t)) \cdot \gamma'(t)}_{\text{Chain Rule}} = 0.$$

Evaluating the above expression at $t = t_0$, we have:

$$\nabla F(\underbrace{\gamma(t_0)}_{P}) \cdot \gamma'(t_0) = 0$$

(Note that $\nabla F(P)$ and $\gamma'(t_0)$ are both vectors in $\mathbb{R}^n$.)

Let F be a function in 3 variables. Let $P_0 = (x_0, y_{0}, _0)$ be a fixed point on the level surface $F(x, y, z) = c$ (Hence, $F(P_0) = c$). If $\nabla F(P_0)$ is defined and nonzero, the **tangent plane** to the surface $F(x, y, z) = c$ at P_0 is defined to be the plane corresponding to the equation:

$$F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0,$$

or more concisely:

$$\nabla F(P_0) \cdot \overrightarrow{P_0 P} = 0, \quad P = (x, y, z).$$

In particular, $\vec{n} = \nabla F(P_0)$ is a normal vector to the tangent plane at P_0.

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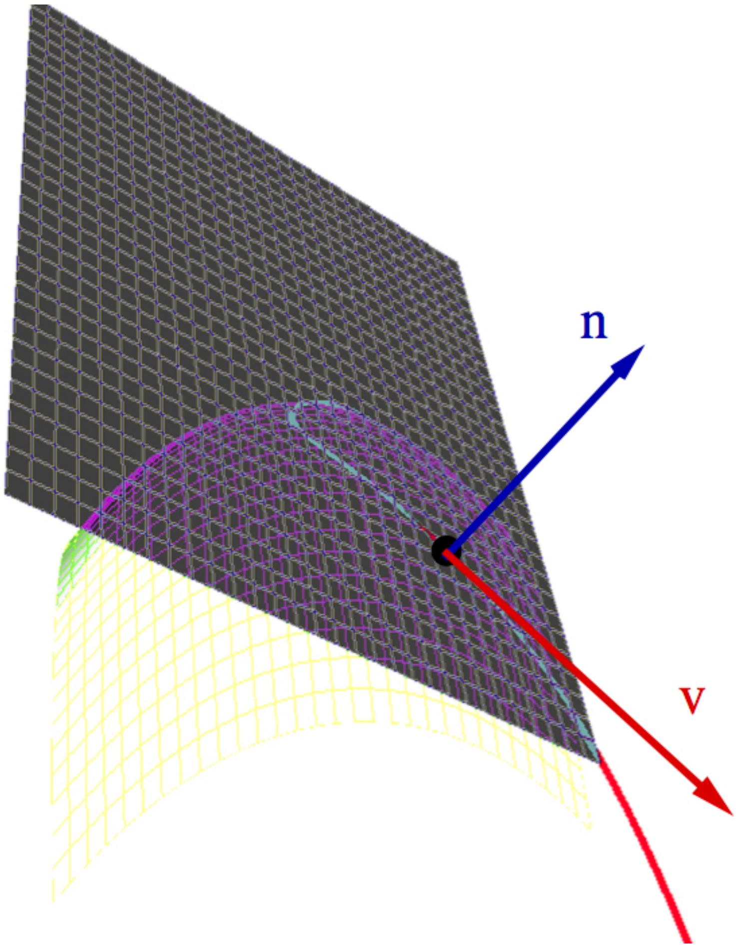


Example.

For the level surface $F(x,y,z)=x^2+4y^2+z^2=4$, the tangent plane to the surface at $P_0=(\sqrt{2}/2,\sqrt{2}/2,\sqrt{6}/2)$ corresponds to the equation:

$$\sqrt{2}(x-\sqrt{2}/2)+4\sqrt{2}(y-\sqrt{2}/2)+\sqrt{6}(z-\sqrt{6}/2)=0.$$

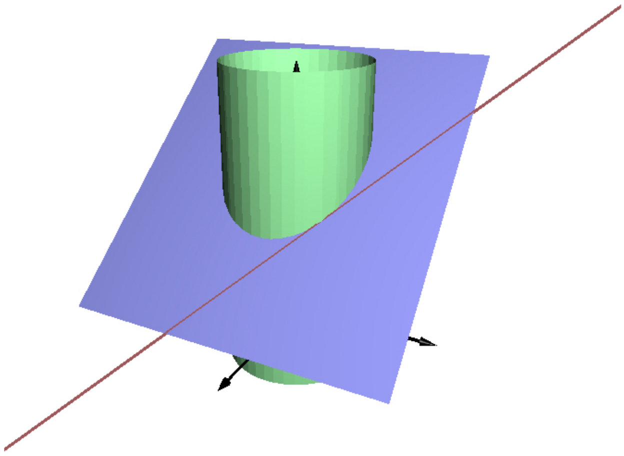
Example.

Let E be the curve which is the intersection of the surfaces:

$$x^2 + y^2 - 2 = 0$$

$$x + z - 4 = 0$$

Find a vector parameterization for the line which is tangent to the curve E at the point $P_0 = (1, 1, 3)$.



Taylor's Theorem for Functions in Two Variables

Let $f(x, y)$ be a function in two variables, $n \in \mathbb{N}$. Suppose the partial derivatives of f of all orders up to $n + 1$ exist and are continuous at all points in an open ball B of positive radius centred at (a, b), then for $(x, y) \in B$, we have:

$$f(x, y) = p_n(x, y) + +R_n(x, y),$$

where:

$$p_n(x, y) = \sum_{k=0}^{n} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \frac{\partial^k f}{\partial x^{k-j} \partial y^j}\bigg|_{(a,b)} (x - a)^{k-j} (y - b)^j$$

$$= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$
$$+ \frac{1}{2!}\left(f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2\right)$$
$$+ \frac{1}{3!}\left(f_{xxx}(a, b)(x - a)^3 + 3f_{xxy}(a, b)(x - a)^2(y - b)\right.$$
$$\left.+3f_{xyy}(a, b)(x - a)(y - b)^2 + f_{yyy}(a, b)(y - b)^3\right) + \cdots,$$

and:

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$$R_n(x,y) = \frac{1}{(n+1)!}\sum_{j=0}^{n+1}\binom{n+1}{j}\frac{\partial^{n+1} f}{\partial x^{n+1-j}\partial y^j}\bigg|_{(a+c(x-a),b+c(x-b))}(x-a)^{n+1-j}(y-b)^j,$$

for some $c \in (0,1)$.
The polynomial $p_n(x,y)$ is called the **n-th Taylor Polynomial** of $f(x,y)$ about (a,b).

Example.

Let $f(x,y) = \sin x \sin y$. Approximate the value of $f(0.01,-0.2)$ using the second Taylor Polynomial of f about $(0,0)$. We have:

$$f_x(x,y) = \cos x \sin y, \quad f_y(x,y) = \sin x \cos y,$$

$$f_{xx}(x,y) = -\sin x \sin y, \quad f_{xy}(x,y) = \cos x \cos y, \quad f_{yy}(x,y) = -\sin x \sin y.$$

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Hence, the second Taylor Polynomial of f about $(0,0)$ is:

$$\begin{aligned} p(x,y) &= f(0,0) + f_x(0,0)x + f_y(0,0)y \\ &\quad + \frac{1}{2!}\left(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right) \\ &= 0 + 0 + 0 + \frac{1}{2!}(0 + 2\cdot 1\cdot xy + 0) = xy. \end{aligned}$$

So, $f(0.01,-0.2)$ is approximately equal to $p(0.01,-0.2) = (0.01)(-0.2) = -0.002$.

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The error of the approximation is:

$$\begin{aligned} &|f(0.01,-0.2) - p(0.01,-0.2)| = |R_3(0.01,-0.2)| \\ &= \left|\frac{1}{3!}\left(f_{xxx}(0.01c,-0.2c)(0.01)^3 + 3f_{xxy}(0.01c,-0.2c)(0.01)^2(-0.2)\right.\right. \\ &\quad \left.\left. +3f_{xyy}(0.01c,-0.2c)(0.01)(-0.2)^2 + f_{yyy}(0.01c,-0.2c)(-0.2)^3\right)\right|, \end{aligned}$$

for some $c \in (0,1)$.

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Computing the 3-rd order partial derivatives of f, we have:

$$\begin{aligned} &|R_3(0.01,-0.2)| \\ &= \left|\frac{1}{3!}\left(-\cos(0.01c)\sin(-0.2c)(0.01)^3 - 3\sin(-0.01c)\cos(-0.2c)(0.01)^2(-0.2)\right.\right. \\ &\quad \left.\left. -3\cos(0.01c)\sin(-0.2c)(0.01)(-0.2)^2 - \sin(0.01c)\cos(-0.2c)(-0.2)^3\right)\right| \\ &\le \frac{1}{6}\left(|0.01|^3 + 3|0.01|^2|-0.2| + 3|0.01||-0.2|^2 + |-0.2|^3\right), \end{aligned}$$

since the sine and cosine functions have absolute values less than or equal to 1.