Week 10
The Gradient

Tangent Vector to a Parametric Curve

Let $\vec{r}(t) = \langle r_1(t), r_2(t), \ldots, r_n(t) \rangle$, $t \in \mathbb{R}$, be a parametric curve in $\mathbb{R}^n$. The tangent vector to $\vec{r}$ at $t = t_0$ is by definition:

$$\vec{v} = \vec{r}\,'(t_0) := \langle r_1'(t_0), r_2'(t_0), \ldots, r_n'(t_0) \rangle.$$

The Gradient Vector

Definition.

Let F be a function in n variables $x_1, x_2, \ldots, x_n$. The gradient of F at $P = (a_1, a_2, \ldots, a_n)$ is the vector:

$$\langle F_{x_1}(P), F_{x_2}(P), \ldots, F_{x_n}(P) \rangle \in \mathbb{R}^n.$$

Here,

$$F_{x_i}(P) = \left.\frac{\partial F}{\partial x_i}\right|_{(x_1,x_2,\ldots,x_n)=(a_1,a_2,\ldots,a_n)}$$

Theorem. Let $F(x_1, x_2, \ldots, x_n)$ be a function in n variables, P a point on the level set:

$$F(x_1, x_2, \ldots, x_n) = c$$

If the gradient vector $\nabla F(P) = \langle F_{x_1}(P), F_{x_2}(P), \ldots, F_{x_n}(P) \rangle$ of F at P is nonzero, then $\nabla F(P)$ is perpendicular to the level set $F(x_1, x_2, \ldots, x_n) = c$, in the sense that it is perpendicular to the tangent vector at P to every smooth curve on $F(x_1, x_2, \ldots, x_n) = c$ which passes through P.
In other words:

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Claim.

If I is an open interval in $\mathbb{R}$, and a differentiable vector-valued function $\gamma : I \to \mathbb{R}^n$ satisfies:

$$F(\gamma(t)) = c \quad \text{(i.e. The curve lies on the level set.)}$$
$$\gamma(t_0) = P, \quad t_0 \in I, \quad \text{(i.e. The curve passes through the point } P \text{ when } t = t_0\text{)}$$

then:

$$\nabla F(P) \cdot \gamma'(t_0) = 0.$$

Proof.

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Suppose $\gamma(t) = \langle \gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t) \rangle$, where γ_i is a differentiable real-valued function in one variable. Applying $\frac{d}{dt}$ to both sides of $F(\gamma(t)) = c$, we have:

$$\frac{d}{dt} F(\gamma(t)) = \frac{d}{dt} c$$
$$\underbrace{\nabla F(\gamma(t)) \cdot \gamma'(t)}_{\text{Chain Rule}} = 0.$$

Evaluating the above expression at $t = t_0$, we have:

$$\nabla F(\underbrace{\gamma(t_0)}_{P}) \cdot \gamma'(t_0) = 0$$

(Note that $\nabla F(P)$ and $\gamma'(t_0)$ are both vectors in $\mathbb{R}^n$.)

Example.

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Consider the level curve $F(x, y) = x^2 + y^2 = 4$ in $\mathbb{R}^2$. Let $\gamma(t) = \langle 2\cos(t), 2\sin(t) \rangle$, $t \in \mathbb{R}$. Then,

$$F(\gamma(t)) = (2\cos(t))^2 + (2\sin(t))^2 = 4,$$

so the curve γ lies on $F(x, y) = 4$. (In fact, in this case γ *is* the entire level curve.) Let $P = (\sqrt{3}, 1) = \gamma(\pi/6)$ on the level curve. Since $\nabla F = \langle 2x, 2y \rangle$, $\gamma'(t) = \langle -2\sin(t), 2\cos(t) \rangle$, we have:

$$\nabla F(P) \cdot \gamma'(\pi/6) = \langle 2\sqrt{3}, 2 \rangle \cdot \left\langle -2\left(\frac{1}{2}\right), 2\left(\frac{\sqrt{3}}{2}\right) \right\rangle = -2\sqrt{3} + 2\sqrt{3} = 0$$

Hence, the vectors $\nabla(F)(P)$ and $\gamma'(\pi/6)$ are perpendicular to each other.

Example.

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Consider the level surface $F(x, y, z) = x^2 + 4y^2 + z^2 = 4$, curve $\gamma(t) = \langle \sin t, \cos t, \sqrt{3}\sin t\rangle$, $t \in \mathbb{R}$, and $P = (\sqrt{2}/2, \sqrt{2}/2, \sqrt{6}/2) = \gamma(\pi/4)$ on the surface. (Note that $F(\gamma(t)) = \sin^2 t + 4\cos^2 t + (\sqrt{3}\sin t)^2 = 4$, so the curve does lie on the surface.) Then, $\nabla F = \langle 2x, 8y, 2z\rangle$, $\gamma'(t) = \langle \cos t, -\sin t, \sqrt{3}\cos t\rangle$. Hence,

$$\nabla F(P) \cdot \gamma'(\pi/4) = \langle \sqrt{2}, 4\sqrt{2}, \sqrt{6}\rangle \cdot \langle \sqrt{2}/2, -\sqrt{2}/2, \sqrt{6}/2\rangle = 1 - 4 + 3 = 0.$$

Let F be a function in 3 variables. Let $P_0 = (x_0, y_{0,0})$ be a fixed point on the level surface $F(x, y, z) = c$ (Hence, $F(P_0) = c$). If $\nabla F(P_0)$ is defined and nonzero, the **tangent plane** to the surface $F(x, y, z) = c$ at P_0 is defined to be the plane corresponding to the equation:

$$F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0,$$

or more concisely:

$$\nabla F(P_0) \cdot \overrightarrow{P_0 P} = 0, \quad P = (x, y, z).$$

In particular, $\vec{n} = \nabla F(P_0)$ is a normal vector to the tangent plane at P_0.

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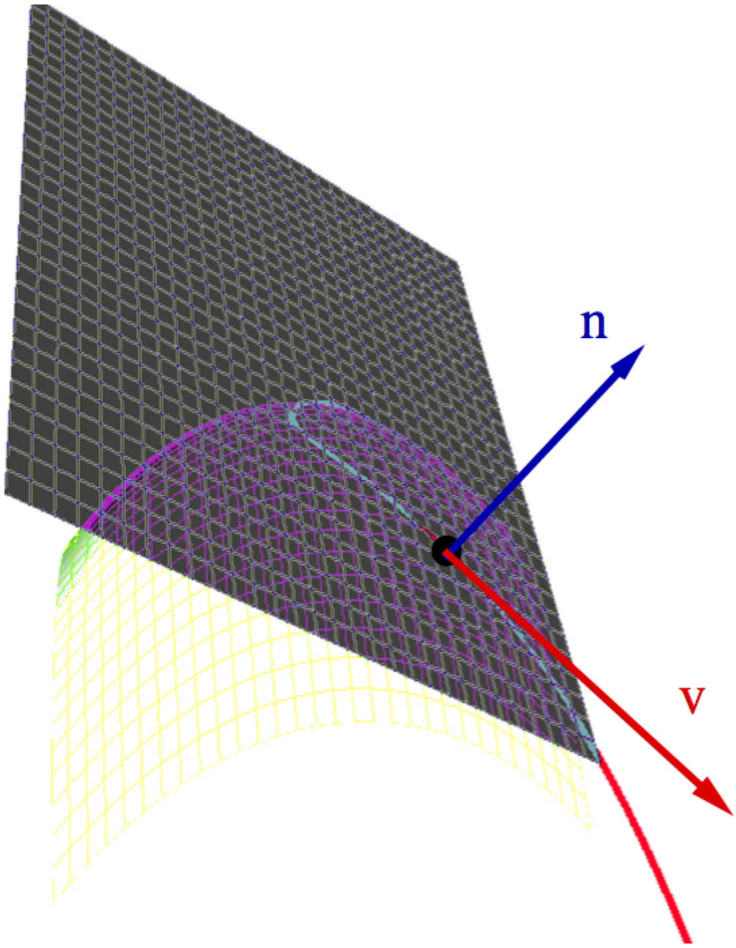

Example.

For the level surface $F(x,y,z) = x^2 + 4y^2 + z^2 = 4$, the tangent plane to the surface at $P_0 = (\sqrt{2}/2, \sqrt{2}/2, \sqrt{6}/2)$ corresponds to the equation:

$$\sqrt{2}(x - \sqrt{2}/2) + 4\sqrt{2}(y - \sqrt{2}/2) + \sqrt{6}(z - \sqrt{6}/2) = 0.$$