THE CHINESE UNIVERSITY OF HONG KONG
MATH 1540 Homework Set 2
Due time 6:30 pm Oct 13, 2016

1. Find the determinants of the following matrices:

   (a)
   $$\begin{pmatrix} 10 & -1 \\ 1 & -2 \end{pmatrix}$$

   (b)
   $$\begin{pmatrix} 6 & -3 & 3 \\ 0 & 2 & 7 \\ -9 & 5 & 4 \end{pmatrix}$$

   (c)
   $$\begin{pmatrix} 20 & 7 & 13 & 5 \\ 0 & 6 & -8 & 5 \\ 12 & 1 & 15 & 5 \\ 0 & 0 & 6 & 11 \end{pmatrix}$$

2. (a) Let A be an $n \times n$ square matrix, λ a real number. Show that there exists a nonzero $\vec{v} \in \mathbb{R}^n$ such that:
   $$A\vec{v} = \lambda\vec{v}$$
   if and only if $\det(A - \lambda I) = 0$.
   (Here, I is the $n \times n$ identity matrix.)

   (b) Find all $\lambda \in \mathbb{R}$ such that:
   $$\begin{pmatrix} 1 & 0 & -3 \\ 0 & -2 & 0 \\ -3 & 0 & 1 \end{pmatrix} \vec{x} = \lambda\vec{x}$$
   has a nonzero solution $\vec{x} \in \mathbb{R}^n$.
   (Such λ's are called *eigenvalues* of the matrix A.)

3. Determine if each of the following matrices is singular (i.e. non-invertible), either via Gaussian elimination or by computing its determinant.

   (a)
   $$\begin{pmatrix} 1 & 0 & -4 \\ 7 & 4 & 6 \\ 3 & -5 & -2 \end{pmatrix}$$

   (b)
   $$\begin{pmatrix} 0 & 5 & 7 & 3 \\ 6 & -3 & 0 & 0 \\ 8 & 3 & -7 & -7 \\ -5 & -5 & 2 & -6 \end{pmatrix}$$

(c)
$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -2 & -2 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & -2 & 1 & 3 \end{pmatrix}$$

4. Let:

$$A = \begin{pmatrix} -1 & 3 & 0 \\ 0 & 2 & -1 \\ 4 & 3 & 2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 16 \\ 8 \\ 15 \end{pmatrix}.$$

Given that A is invertible, solve the following matrix equation:

$$A\vec{x} = \vec{b}$$

using:

(a) Cramer's Rule.

(b) Gaussian elimination on the augmented matrix $\left(\; A \mid \vec{b} \; \right)$.

(c) $\vec{x} = A^{-1}\vec{b}$, where A^{-1} is obtained by performing Gaussian elimination on $(\; A \mid I \;)$.

5. Show that in general $\det(A + B) \neq \det A + \det B$.

Linear Independence

Definition. We say that a set of vectors:

$$\vec{v}_1, \vec{v}_2, \ldots \vec{v}_m$$

in $\mathbb{R}^n$ are **linearly independent** if the only solution to

$$x_1 v_1 + x_2 v_2 + \cdots x_m v_m = \vec{0}$$

is $x_1 = x_2 = \cdots = x_m = 0$.

This is equivalent to saying that the matrix equation:

$$\underbrace{\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{pmatrix}}_{A} \vec{x} = \vec{0}$$

has $\vec{x} = \vec{0}$ as its only solution.

Examining the augmented matrix $\left(\begin{array}{c|c} A & \vec{0} \end{array} \right)$, we see that $A\vec{x} = \vec{0}$ has a unique solution $\vec{x} = 0$ if and only if A is row equivalent to a matrix of the form:

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In particular, if $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent, then $m \leq n$.

Example. Determine if the following vectors are linearly independent:

$$\left\{ \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 0 \\ -3 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 4 \end{pmatrix}, \vec{v}_4 = \begin{pmatrix} 3 \\ 6 \\ 2 \\ 5 \end{pmatrix} \right\}.$$

Let

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 \\ -1 & 5 & 1 & 6 \\ 2 & 0 & 0 & 2 \\ 0 & -3 & 4 & 5 \end{pmatrix}$$

be the matrix whose columns are the $\vec{v}_i$'s.

Applying Gaussian elimination, we see that A is row equivalent to:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

What this tells us is that the solutions to the equation:

$$x_1\vec{v}_1 + \cdots + x_4\vec{v}_4 = \vec{0}$$

are precisely those to the equation:

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} = \vec{0},$$

which has a solution where not all x_i's are zero. For example,

$$x_1 = 1, x_2 = 1, x_3 = 2, x_4 = -1.$$

Hence, the vectors $\vec{v}_1, \dots \vec{v}_4$ are not linearly independent.

Note also from the row reduced augmented matrix that the smaller matrix formed by $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is row equivalent to:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which implies that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linear independent.

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6. Determine if each of the following sets of vectors in $\mathbb{R}^3$ are linearly independent:

   (a)

$$\left\{ \begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \right\}$$

   (b)

$$\left\{ \begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ -7 \end{pmatrix} \right\}$$

   (c)

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ -7 \end{pmatrix} \right\}$$

7. Show that if three vectos v_1, v_2, v_3 in $\mathbb{R}^3$ are linearly independent, then every vector $\vec{v} \in \mathbb{R}^3$ may be expressed uniquely as a linear combination of v_1, v_2, v_3. In other words, there are unique scalars $\lambda_1, \lambda_2, \lambda_3$ such that:

$$\vec{v} = \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \lambda_3 \vec{v}_3.$$