# Construction of Preconditioners for Wiener-Hopf Equations by Operator Splitting*

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**Abstract**

In this paper, we propose a new type of preconditioners for solving finite section Wiener-Hopf integral equations $(\alpha\mathcal{I}+\mathcal{A}_\tau)x_\tau = g$ by the preconditioned conjugate gradient algorithm. We show that for an integer $u > 1$, the operator $\alpha\mathcal{I}+\mathcal{A}_\tau$ can be decomposed into a sum of operators $\alpha\mathcal{I}+\mathcal{P}_\tau^{(u,v)}$ for $0 \le v < u$. Here $\mathcal{P}_\tau^{(u,v)}$ are $\{\omega_v\}$-circulant integral operators that are the continuous analog of $\{\omega_v\}$-circulant matrices. For $u \ge 1$, our preconditioners are defined as $(1/u)\sum_v(\alpha\mathcal{I}+\mathcal{P}_\tau^{(u,v)})^{-1}$. Thus the way the preconditioners are constructed is very similar to the approach used in the additive Schwarz method for elliptic problems. As for the convergence rate, we prove that the spectra of the resulting preconditioned operators $[(1/u)\sum_v(\alpha\mathcal{I}+\mathcal{P}_\tau^{(u,v)})^{-1}][\alpha\mathcal{I}+\mathcal{A}_\tau]$ are clustered around 1 and thus the algorithm converges sufficiently fast. Finally, we discretize the resulting preconditioned equations by rectangular rule. Numerical results show that our methods converges faster than those preconditioned by using circulant integral operators.

**Abbreviated Title:** Splitting of Wiener-Hopf Integral Operators.

**Key Words.** Wiener-Hopf integral operator, projection method, preconditioned conjugate gradient method, $\{\omega_v\}$-circulant integral operator, Fourier transform, rectangular rule.

**AMS(MOS) Subject Classifications.** 45E10, 45L10, 65R20, 65J10.

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# 1 Introduction

Wiener-Hopf integral equations arise in a variety of practical applications in mathematics and engineering especially in the solutions of inverse problems. Typical examples are linear prediction problems for stationary stochastic processes [10, pp.145-146], the diffusion problems [10, pp.186-187], the scattering problems [10, pp.188-189], the distribution of temperature in a stellar atmosphere in a radioactive equilibrium [12, p.195]. In [15, pp.264–265], it has also been mentioned some important problems for Wiener-Hopf integral equations. In this paper, we consider the solutions of Wiener-Hopf integral equations on the half-line $[0,\infty)$. Let $a(t) \in L_1(-\infty,\infty)$ be a conjugate symmetric complex-valued function, i.e. $\overline{a(t)} = a(-t)$, and

$$(\mathcal{A}x)(t) = \int_0^\infty a(t-s)x(s)ds, \quad 0 \le t < \infty, \tag{1.1}$$

be the associated convolution operator acting on the Hilbert space $L_2[0,\infty)$. The half-line Wiener-Hopf equation concerned is given as follows:

$$\alpha x(t) + \mathcal{A}x(t) = g(t), \quad 0 \le t < \infty, \tag{1.2}$$

where $\alpha > 0$ and $g(t)$ is a given function in $L_2[0,\infty)$. In practical applications, the positive constant $\alpha$ is served as a Tikhonov regularization parameter, see Kress [13, p.243] for instance.

In contrast to the whole line Wiener-Hopf equation, we remark that the half-line Wiener-Hopf equation cannot be solved explicitly by using Fourier transform. One numerical method of approximating the solution $x(t)$ is by the projection method [7, Chapter 3]. The solution $x(t)$ of the Wiener-Hopf equation is approximated by $\tilde{x}(t)$:

$$\tilde{x}(t) = \begin{cases} x_\tau(t), & 0 \le t \le \tau, \\ 0, & t > \tau. \end{cases}$$

Here $x_\tau(t)$ is the solution of the following finite section Wiener-Holf equation

$$\alpha x_\tau(t) + \mathcal{A}_\tau x_\tau(t) = g(t), \quad 0 \le t \le \tau, \tag{1.3}$$

with $\mathcal{A}_\tau$ given by

$$(\mathcal{A}_\tau x_\tau)(t) = \int_0^\tau a(t-s)x_\tau(s)ds, \quad 0 \le t \le \tau. \tag{1.4}$$

It has been proved that the approximated solution $\tilde{x}(t)$ converges to $x(t)$ in the $L_2$-norm of the Hilbert space $L_2[0,\infty)$ as $\tau$ tends to infinity [7, Theorem 3.1].

Recently, Gohberg, Hanke and Koltracht [9] employed the conjugate gradient method as an iterative method for solving the finite section of Wiener-Hopf equations. Although

$\mathcal{A}_\tau$ is a compact operator, the spectrum of $\mathcal{A}_\tau$ becomes dense in the spectrum of $\mathcal{A}$ as $\tau$ tends to infinity, see Gohberg *et. al.* [9]. Thus the convergence rate of the conjugate gradient method will not be superlinear for sufficiently large $\tau$. In order to speed up the convergence rate of the method, they used *circulant integral operators* $\mathcal{G}_\tau$ to precondition $\mathcal{A}_\tau$. Circulant integral operators are operators of the form

$$(\mathcal{G}_\tau y)(t) = \int_0^\tau g_\tau(t-s)y(s)ds, \quad 0 \le t \le \tau,$$

where $g_\tau$ is a $\tau$-periodic conjugate symmetric function in $L_1[-\tau, \tau]$. In this case, one solves the preconditioned equation

$$[(\sigma\mathcal{I} + \mathcal{G}_\tau)^{-1}(\sigma\mathcal{I} + \mathcal{A}_\tau)x_\tau](t) = [(\sigma\mathcal{I} + \mathcal{G}_\tau)^{-1}g](t), \quad 0 \le t \le \tau,$$

where $\mathcal{I}$ is an identity operator. A unifying approach of constructing circulant integral operators as preconditioners for solving Wiener-Hopf equations is given in Chan, Jin and Ng [3]. It has been shown that the spectra of these circulant preconditioned operators are clustered around 1. Hence the preconditioned conjugate gradient method converges superlinearly even for sufficiently large $\tau$.

In this paper, we propose a new type of operators as preconditioners for solving finite section of Wiener-Hopf equations (1.3). Using the analog result in $\{\omega\}$-circulant matrices [5, p.74], we define $\{\omega\}$-circulant integral operator in §2. Then we prove that for an integer $u > 1$, the convolution operator $\mathcal{A}_\tau$ can be written as a sum of $\{\omega_v\}$-circulant integral operators $\mathcal{P}_\tau^{(u,v)}$ with $\omega_v = e^{2\pi i v/u}$, i.e.

$$\mathcal{A}_\tau = \frac{1}{u}\sum_{v=0}^{u-1} \mathcal{P}_\tau^{(u,v)}.$$

Therefore, $\alpha\mathcal{I} + \mathcal{A}_\tau$ can be written as

$$\alpha\mathcal{I} + \mathcal{A}_\tau = \frac{1}{u}\sum_{v=0}^{u-1} \left(\alpha\mathcal{I} + \mathcal{P}_\tau^{(u,v)}\right).$$

For an integer $u \ge 1$, our proposed preconditioners are defined to be the operator $\mathcal{B}_\tau^{(u)}$ given by

$$\mathcal{B}_\tau^{(u)} = \frac{1}{u}\sum_{v=0}^{u-1} \left(\alpha\mathcal{I} + \mathcal{P}_\tau^{(u,v)}\right)^{-1},$$

if all the operators $\alpha\mathcal{I} + \mathcal{P}_\tau^{(u,v)}$ are invertible. Instead of solving the original operator equations (1.3), we solve the following preconditioned operator equations

$$[\mathcal{B}_\tau^{(u)}(\alpha\mathcal{I} + \mathcal{A}_\tau)x_\tau](t) = [\mathcal{B}_\tau^{(u)}g](t), \quad 0 \le t \le \tau.$$

We prove that the spectra of the preconditioned operators $(\mathcal{B}_\tau^{(u)})^{1/2}(\alpha\mathcal{I}+\mathcal{A}_\tau)(\mathcal{B}_\tau^{(u)})^{1/2}$ are clustered around 1. It follows that the conjugate gradient method, when applied to solving these preconditioned operator equations, converges superlinearly. Finally, rectangular rule are used to discretize the finite section of Wiener-Hopf and $\{\omega_v\}$-circulant integral operators. The discretization matrices are Toeplitz matrices and $\{\omega_v\}$-circulant matrices respectively. Numerical results show that our preconditioners perform better than those of using circulant integral operators proposed by Gohberg *et. al.* [9].

The outline of the paper is as follows. In §2, we define $\{\omega\}$-circulant integral operators and decompose $\mathcal{A}_\tau$ into these circulant integral operators. We then construct our preconditioners. In §3, we prove that the spectra of these preconditioned operators are clustered around 1. In §4, we use rectangular rule to discretize the integral operators and some numerical examples are presented to illustrate the convergence performance of the algorithm. Some concluding remarks are given in §5.

# 2 Decomposition of $\mathcal{A}_\tau$ and Construction of Preconditioners

In this section, we consider some properties of the convolution operator $\mathcal{A}_\tau$. These properties are useful in the construction of the preconditioners. Before we begin, let us first recall that the Fourier transform of a function $q(t)$ is defined by

$$\hat{q}(t) \equiv \int_{-\infty}^{\infty} q(s)e^{-ist}ds, \quad \forall t \in \mathbf{R}$$

and the convolution product $\hat{p} * \hat{q}$ of $p(t)$ with $q(t)$ is defined by

$$(\hat{p} * \hat{q})(t) \equiv \int_{-\infty}^{\infty} p(s)q(s)e^{-ist}ds, \quad \forall t \in \mathbf{R},$$

where $\hat{p}(t)$ and $\hat{q}(t)$ are the Fourier transforms of $p(t)$ and $q(t)$ respectively. In addition, the inner product in the Hilbert space $L_2[0,\tau]$ is defined by

$$\langle p, q\rangle \equiv \int_0^\tau p(t)\overline{q(t)}dt.$$

In Grenander and Szegö [11, p.139], it has been shown that the eigenvalues of $\mathcal{A}_\tau$ have asymptotically the distribution of the values of $\hat{a}(t)$. In particular, we have the following Lemma.

**LEMMA 1** *Let $a(t) \in L_1(-\infty,\infty)$. Then the spectrum of operator $\mathcal{A}_\tau$ satisfies*

$$\sigma(\mathcal{A}_\tau) \subseteq [m, M], \quad \forall \tau > 0,$$

*where $m$ and $M$ are the infimum and supremum of $\hat{a}(t)$ respectively.*

It follows from Lemma 1 that the spectrum $\sigma(\alpha\mathcal{I} + \mathcal{A}_\tau)$ of the finite section Wiener-Hopf integral operator $\alpha\mathcal{I} + \mathcal{A}_\tau$ satisfies

$$\sigma(\alpha\mathcal{I} + \mathcal{A}_\tau) \subseteq [\alpha + m, \alpha + M], \quad \forall \tau > 0.$$

Thus if $\alpha + m > 0$ then $\alpha\mathcal{I} + \mathcal{A}_\tau$ is invertible.

We note that there is a close relationship between the Wiener-Hopf integral operators and semi-infinite Toeplitz matrices, see [7, p.5]. In [4], Chan and Ng proved that given any $n$-by-$n$ Toeplitz matrices $A_n$ and integer $u > 1$, it can be written as a sum of $\{\omega_v\}$-circulant matrices $C_n^{(v)}$, i.e.

$$A_n = \frac{1}{u}\sum_{v=0}^{u-1} C_n^{(v)}.$$

In Davis [5, p.74], it is shown that if $C_n^{(v)}$ is an $n$-by-$n$ $\{\omega_v\}$-circulant matrix, then it has the spectral decomposition

$$C_n^{(v)} = D_n F_n \Lambda_n F_n^* D_n^*.$$

Here $F_n$ is the Fourier matrix with entries

$$[F_n]_{k,j} = \frac{1}{\sqrt{n}} e^{2\pi ijk/n},$$

and $D_n$ is a diagonal matrix given by

$$D_n = \mathrm{diag}[1, \omega_v^{1/n}, \cdots, \omega_v^{(n-1)/n}]$$

and $\Lambda_n$ is a diagonal matrix holding the eigenvalues of $C_n^{(v)}$. Similar to the discrete case, we show below that the convolution operator $\mathcal{A}_\tau$ can be written as a sum of $\{\omega\}$-circulant integral operators which are defined as follows:

**DEFINITION** Let $\omega = e^{i\theta_0}$ with $\theta_0 \in [0, 2\pi)$. An operator $\mathcal{C}_\tau$ is a $\{\omega\}$-circulant integral operator if it is of the form

$$(\mathcal{C}_\tau y)(t) = \int_0^\tau c_\tau(t-s) y(s) ds, \quad 0 \le t \le \tau. \tag{2.1}$$

where $c_\tau$ is a conjugate symmetric function in $L_1[-\tau, \tau]$ with

$$c_\tau(t) = e^{-i\theta_0} c_\tau(t+\tau), \quad -\tau \le t \le 0. \tag{2.2}$$

We remark that $\{1\}$-circulant integral operators are just the circulant integral operators used by Gohberg *et. al.* [9] in solving the finite section of Wiener-Hopf integral equations. The following Lemma is about the eigen-decomposition of the $\{\omega\}$-circulant integral operator $\mathcal{C}_\tau$.

**LEMMA 2** *Let $\mathcal{C}_\tau$ be a $\{\omega\}$-circulant integral operator defined as in (2.1) with $\omega = e^{i\theta_0}$. Then $\mathcal{C}_\tau$ is a compact self-adjoint operator on $L_2[0,\tau]$ and its complete set of eigenfunctions is given by*

$$\{\phi_n^{(\theta_0)}(t) \quad | \quad \phi_n^{(\theta_0)}(t) = \frac{1}{\sqrt{\tau}} e^{2\pi int/\tau} \cdot e^{i\theta_0 t/\tau}, \ n \in \mathbf{Z}\},$$

*where $\mathbf{Z}$ is the set of all integers. Furthermore, the eigenvalues $\lambda_n(\mathcal{C}_\tau)$ of the $\{\omega\}$-circulant operator $\mathcal{C}_\tau$ are given by*

$$\lambda_n(\mathcal{C}_\tau) = \int_0^\tau c_\tau(t) e^{-2\pi int/\tau} \cdot e^{-i\theta_0 t/\tau} dt = \sqrt{\tau}\langle c_\tau, \phi_n^{(\theta_0)}\rangle, \quad \forall n \in \mathbf{Z}. \tag{2.3}$$

**PROOF:** Since $c_\tau$ is a conjugate symmetric function in $L_1[-\tau,\tau]$, $\mathcal{C}_\tau$ is a self-adjoint operator acting on $L_2[0,\tau]$. Taking

$$\phi_n^{(\theta_0)}(t) = \frac{1}{\sqrt{\tau}} e^{2\pi int/\tau} \cdot e^{i\theta_0 t/\tau}, \quad \forall n \in \mathbf{Z}$$

as orthonormal basis for $L_2[0,\tau]$, it follows from the periodicity of $c_\tau(t)$ mentioned in (2.2) and the definition of $\phi_n^{(\theta_0)}(t)$ that we obtain

$$\begin{aligned}
(C_\tau \phi_n^{(\theta_0)})(t) &= \int_0^\tau c_\tau(t-s)\phi_n^{(\theta_0)}(s) ds = \sqrt{\tau}\left(\int_{t-\tau}^t c_\tau(s)\phi_n^{(\theta_0)}(-s) ds\right) \cdot \phi_n^{(\theta_0)}(t) \\
&= \sqrt{\tau}\left(\int_{t-\tau}^0 e^{-i\omega} c_\tau(s)\phi_n^{(\theta_0)}(-s) ds + \int_0^t c_\tau(s)\phi_n^{(\theta_0)}(-s) ds\right) \cdot \phi_n^{(\theta_0)}(t) \\
&= \sqrt{\tau}\left(\int_0^\tau c_\tau(s)\phi_n^{(\theta_0)}(-s) ds\right) \cdot \phi_n^{(\theta_0)}(t).
\end{aligned}$$

By the definition of eigenvalues and eigenvectors of an operator (see [8, p.108] for instance), the results of the lemma follow. □

Then we have the following main theorem about the decomposition of the convolution operator $\mathcal{A}_\tau$.

**THEOREM 1** *For an integer $u > 1$, then we have*

$$\mathcal{A}_\tau = \frac{1}{u}\sum_{v=0}^{u-1} \mathcal{P}_\tau^{(u,v)}, \tag{2.4}$$

*where $\mathcal{P}_\tau^{(u,v)}$ are $\{\omega_v\}$-circulant integral operators with $\omega_v = e^{2\pi iv/u}$ and its eigenvalues are given by*

$$\lambda_n(\mathcal{P}_\tau^{(u,v)}) = (\hat{D}_\tau * \hat{a})(\frac{2\pi n}{\tau} + \frac{2\pi v}{u\tau}), \quad \forall n \in \mathbf{Z}, \tag{2.5}$$

*where $D_\tau$ is the function given by*

$$D_\tau(t) = \begin{cases} 1, & |t| \le \tau, \\ 0, & |t| > \tau. \end{cases} \tag{2.6}$$

**PROOF:** Letting

$$p_\tau^{(u,v)}(t) = \begin{cases} D_\tau(t)a(t) + e^{2\pi iv/u}D_\tau(t-\tau)a(t-\tau), & 0 \le t \le \tau, \\ e^{-2\pi iv/u}D_\tau(t+\tau)a(t+\tau) + D_\tau(t)a(t), & -\tau \le t \le 0. \end{cases} \tag{2.7}$$

As $a(t) \in L_1(-\infty,\infty)$, the kernel function $p_\tau^{(u,v)}$ is in $L_1[-\tau,\tau]$. It is straightforward to show that the convolution operator $\mathcal{P}_\tau^{(u,v)}$ with kernel function $p_\tau^{(u,v)}(t)$ is an $\{\omega_v\}$-circulant integral operator with $\omega_v = e^{2\pi iv/u}$ . Since $\sum_{v=0}^{u-1} e^{2\pi iv/u} = 0$, we have

$$\frac{1}{u}\sum_{v=0}^{u-1} p_\tau^{(u,v)}(t) = D_\tau(t)a(t) = a(t), \quad -\tau \le t \le \tau.$$

Since both $D_\tau(t)$ and $a(t)$ are conjugate symmetric, (2.4) follows. By (2.3), the eigenvalues of the operator $\mathcal{P}_\tau^{(u,v)}$ are given by

$$\lambda_n(\mathcal{P}_\tau^{(u,v)}) = \int_0^\tau p_\tau^{(u,v)}(t)e^{-2\pi int/\tau} \cdot e^{-2\pi ivt/\tau u}dt, \quad \forall n \in \mathbf{Z}.$$

By (2.7) and noting that the support of $D_\tau$ is contained in $[-\tau,\tau]$, the eigenvalues of the operator $\mathcal{P}_\tau$ are given by

$$\lambda_n(\mathcal{P}_\tau^{(u,v)}) = \int_{-\tau}^\tau D_\tau(t)a(t)e^{-2\pi int/\tau} \cdot e^{-2\pi ivt/\tau u}dt = (\hat{D}_\tau * \hat{a})(\frac{2\pi n}{\tau} + \frac{2\pi v}{u\tau}), \quad \forall n \in \mathbf{Z}. \quad \square$$

It follows from Theorem 1 that the operator $\alpha\mathcal{I}+\mathcal{A}_\tau$ can be decomposed into a sum of integral operators, i.e.

$$\alpha\mathcal{I}+\mathcal{A}_\tau=\frac{1}{u}\sum_{v=0}^{u-1}\left(\alpha\mathcal{I}+\mathcal{P}_\tau^{(u,v)}\right). \tag{2.8}$$

We have the following lemma about the spectral property of each integral operator $\alpha\mathcal{I}+\mathcal{P}_\tau^{(u,v)}$ in the decomposition of $\alpha\mathcal{I}+\mathcal{A}_\tau$.

**LEMMA 3** *Let $a(t)\in L_1(-\infty,\infty)$ and its Fourier transform $\hat{a}(t)$ be non-negative. For any given $\epsilon>0$, Then there exists a $\tau^*>0$ such that for all $\tau>\tau^*$, $0\le v<u$,*

$$\frac{\alpha}{2}\le\lambda_n(\alpha\mathcal{I}+\mathcal{P}_\tau^{(u,v)})\le\rho,\quad\forall n\in\mathbf{Z},$$

*where $\rho$ is a positive constant independent of $\tau$. In particular, the operators $\alpha\mathcal{I}+\mathcal{P}_\tau^{(u,v)}$ are invertible for $0\le v<u$.*

**PROOF:** Since

$$\left|\hat{D}_\tau*\hat{a}(s)-\hat{a}(s)\right|=\left|\int_{|t|\ge\tau}a(t)e^{-its}dt\right|\le\int_{|t|\ge\tau}|a(t)|dt,\quad\forall s\in\mathbf{R},$$

there exists an $\tau^*>0$ such that for all $\tau>\tau^*$,

$$\left|(\hat{D}_\tau*\hat{a}-\hat{a})(\frac{2\pi n}{\tau}+\frac{2\pi v}{\tau u})\right|\le\min\{\frac{\alpha}{2},\frac{M}{2}\},\quad\forall n\in\mathbf{Z}, \tag{2.9}$$

where $M$ is the supremum of $\hat{a}(t)$. Thus by (2.5) and (2.9), we have

$$\begin{aligned}\lambda_n(\alpha\mathcal{I}+\mathcal{P}_\tau^{(u,v)}) &= \alpha+(\hat{D}_\tau*\hat{a}-\hat{a})(\frac{2\pi n}{\tau}+\frac{2\pi v}{\tau u})+\hat{a}(\frac{2\pi n}{\tau}+\frac{2\pi v}{\tau u})\\ &\ge \alpha-(\hat{D}_\tau*\hat{a}-\hat{a})(\frac{2\pi n}{\tau}+\frac{2\pi v}{\tau u})\ge\frac{\alpha}{2},\quad\forall n\in\mathbf{Z}.\end{aligned} \tag{2.10}$$

Similarly, we can derive the upper bound of eigenvalues of $\alpha\mathcal{I}+\mathcal{P}_\tau^{(u,v)}$ and set $\rho=3M/2+\alpha$. $\square$

In the following discussion, we use the decomposition of the finite section Wiener-Hopf integral operator $\sigma\mathcal{I}+\mathcal{A}_\tau$ to construct a new type of preconditioners. The way the preconditioners are constructed is followed from the approach used in the additive Schwarz method for elliptic problems, see Dryia and Widlund [6] for instance. We recall that in the additive Schwarz method, a matrix $A$ is first decomposed into a sum of individual matrices,

$$A = P^{(1)} + P^{(2)} + \cdots + P^{(u)},$$

and then the generalized inverses of these individual matrices are added back together to form a preconditioner $B$ of the original matrix $A$, i.e.

$$B = P^{(1)+} + P^{(2)+} + \cdots + P^{(u)+}.$$

In our case, for any integer $u \geq 1$, our preconditioners $\mathcal{B}_\tau^{(u)}$ are defined to be the operators given by

$$\mathcal{B}_\tau^{(u)} = \frac{1}{u} \sum_{v=0}^{u-1} \left(\alpha \mathcal{I} + \mathcal{P}_\tau^{(u,v)}\right)^{-1}, \tag{2.11}$$

if all the operators $\alpha\mathcal{I} + \mathcal{P}_\tau^{(u,v)}$ are invertible. We recall that $\mathcal{P}_\tau^{(u,v)}$ are $\{\omega_v\}$-circulant integral operators with $\omega_v = e^{2\pi i v/u}$ defined as in Theorem 1. In the following, we give the representation of the preconditioners $\mathcal{B}_\tau^{(u)}$.

**LEMMA 4** *Let $a(t) \in L_1(-\infty,\infty) \bigcap L_2(-\infty,\infty)$. If the operators $\alpha\mathcal{I} + \mathcal{P}_\tau^{(u,v)}$ are invertible for $0 \leq v < u$ and their inverses are given by*

$$\left(\alpha\mathcal{I} + \mathcal{P}_\tau^{(u,v)}\right)^{-1} = \frac{1}{\alpha}\mathcal{I} - \mathcal{Q}_\tau^{(u,v)}, \quad 0 \leq v < u,$$

*where $\mathcal{Q}_\tau^{(u,v)}$ are $\{\omega_v\}$-circulant integral operators given by operators given by*

$$[(\mathcal{Q}_\tau^{(u,v)})y](t) = \sum_{n \in \mathbf{Z}} \frac{\lambda_n(\mathcal{P}_\tau^{(u,v)})}{\alpha[\alpha + \lambda_n(\mathcal{P}_\tau^{(u,v)})]} \langle y, \phi_n^{(2\pi v/u)} \rangle \phi_n^{(2\pi v/u)}(t). \tag{2.12}$$

*In particular, if for $0 \leq v < u$, $\alpha + \lambda_n(\mathcal{P}_\tau^{(u,v)}) > 0$ for all $n \in \mathbf{Z}$, then the eigenvalues of the operator $(1/\alpha)\mathcal{I} - \mathcal{Q}_\tau^{(u,v)}$ are positive.*

**PROOF:** We note from the theorem in [8, Theorem 8.1] that

$$\begin{aligned} \left[\left(\alpha\mathcal{I} + \mathcal{P}_\tau^{(u,v)}\right)^{-1} y\right](t) &= \frac{y(t)}{\alpha} - \sum_{n \in \mathbf{Z}} \frac{\lambda_n(\mathcal{P}_\tau^{(u,v)})}{\alpha[\alpha + \lambda_n(\mathcal{P}_\tau^{(u,v)})]} \langle y, \phi_n^{(2\pi v/u)} \rangle \phi_n^{(2\pi v/u)}(t) \\ &= \left[\left(\frac{1}{\alpha}\mathcal{I} - \mathcal{Q}_\tau^{(u,v)}\right) y\right](t). \end{aligned}$$

As $a(t) \in L_1(-\infty,\infty)\bigcap L_2(-\infty,\infty)$, the kernel function $p_\tau^{(u,v)}$ of $\mathcal{P}_\tau^{(u,v)}$ in (2.7) is in $L_2[-\tau,\tau]$. Using the theorem in [8, Theorem 1.2], we have

$$\sum_{n\in\mathbf{Z}} |\lambda_n(\mathcal{P}_\tau^{(u,v)})|^2 < \infty.$$

It follows that the kernel functions $q_\tau^{(u,v)}$ of $\mathcal{Q}_\tau^{(u,v)}$ given by

$$q_\tau^{(u,v)}(t) = \sum_{n\in\mathbf{Z}} \frac{\lambda_n(\mathcal{P}_\tau^{(u,v)})}{\alpha[\alpha+\lambda_n(\mathcal{P}_\tau^{(u,v)})]} e^{2\pi i(nu+v)t/\tau u}$$

are also in $L_2[-\tau,\tau]$. Then it is straightforward to check that $q_\tau^{(u,v)}$ is conjugate symmetric with

$$q_\tau^{(u,v)}(t) = e^{-2\pi iu/v} q_\tau^{(u,v)}(t+\tau), \quad -\tau \le t \le 0.$$

Therefore, $\mathcal{Q}_\tau^{(u,v)}$ are $\{\omega_v\}$-circulant integral operators. By noting the following equality,

$$\frac{1}{\alpha+\lambda_n(\mathcal{P}_\tau^{(u,v)})} = \frac{1}{\alpha} - \frac{\lambda_n(\mathcal{P}_\tau^{(u,v)})}{\alpha[\alpha+\lambda_n(\mathcal{P}_\tau^{(u,v)})]}, \quad \forall n \in \mathbf{Z},$$

if $\alpha+\lambda_n(\mathcal{P}_\tau^{(u,v)}) > 0$ for all $n \in \mathbf{Z}$, then all the eigenvalues of the operator $(1/\alpha)\mathcal{I} - \mathcal{Q}_\tau^{(u,v)}$ are positive. □

As an application of Lemma 4, we see that the preconditioner $\mathcal{B}_\tau^{(u)}$ is just equal to

$$\frac{1}{u}\sum_{v=0}^{u-1} \left(\alpha\mathcal{I} + \mathcal{P}_\tau^{(u,v)}\right)^{-1} = \frac{1}{\alpha}\mathcal{I} - \frac{1}{u}\sum_{v=0}^{u-1} \mathcal{Q}_\tau^{(u,v)}.$$

Since $\mathcal{Q}_\tau^{(u,v)}$ are themselves convolution operators, the operator $\frac{1}{u}\sum_{v=0}^{u-1}\mathcal{Q}_\tau^{(u,v)}$ is also convolution operator. Thus, we see that $\mathcal{B}_\tau^{(u)}$ are Wiener-Hopf integral operators in general.

# 3 Spectra of Preconditioned Operators

In this section, we study the spectra of the preconditioned Wiener-Hopf operators. We will prove that the preconditioned operators will have clustered spectra. The result is stated as the following Theorem 2.

**THEOREM 2** *Let $a(t) \in L_1(-\infty,\infty)$ and its Fourier transform $\hat{a}(t) \geq 0$. Then for all $\epsilon > 0$, there exist a positive integer $N$ and $\tau^* > 0$ such that for all $\tau > \tau^*$, the spectrum of $(\mathcal{B}_\tau^{(u)})^{1/2}(\alpha\mathcal{I} + \mathcal{A}_\tau)(\mathcal{B}_\tau^{(u)})^{1/2}$ has at most $N$ eigenvalues outside the interval $(1-\epsilon, 1+\epsilon)$.*

Before we prove Theorem 2, we state the following lemma 5 which is useful in the analysis of the spectra of the preconditioned operators.

**LEMMA 5** *Let $a(t) \in L_1(-\infty,\infty)$ and its Fourier transform $\hat{a}(t) \geq 0$. Then for all $\epsilon > 0$, there exist a positive integer $N$ and a $\tau^* > 0$ such that for all $\tau > \tau^*$, there exists a decomposition*

$$\mathcal{A}_\tau - \mathcal{P}_\tau^{(u,0)} = \mathcal{R}_\tau^{(u,0)} + \mathcal{E}_\tau^{(u,0)}$$

*with self-adjoint operators $\mathcal{R}_\tau^{(u,0)}$ and $\mathcal{E}_\tau^{(u,0)}$ satisfying*

$$\text{rank } \mathcal{R}_\tau^{(u,0)} \leq N$$

*and*

$$||\mathcal{E}_\tau^{(u,0)}||_2 \leq \epsilon.$$

*Here $||\cdot||_2$ is the operator norm on the Hilbert space $L_2[0,\tau]$. In particular, the spectrum of $(\alpha\mathcal{I} + \mathcal{P}_\tau^{(u,0)})^{-1/2}(\alpha\mathcal{I} + \mathcal{A}_\tau)(\alpha\mathcal{I} + \mathcal{P}_\tau^{(u,0)})^{-1/2}$ has at most $N$ eigenvalues outside the interval $(1-\epsilon, 1+\epsilon)$.*

**PROOF:** The main point is that $\hat{D}_\tau * \hat{a}$ converges to $\hat{a}$ uniformly on $\mathbf{R}$. Then the proof of this Lemma can be followed from Theorems 1 and 3 in Chan *et. al.* [3].

**LEMMA 6** *Let $a(t) \in L_1(-\infty,\infty)$. Let $\mathcal{K}_\tau$ and $\mathcal{H}_\tau$ be two self-adjoint convolution operators with kernel functions given by*

$$k_\tau(t) = D_\tau(t)a(t), \quad -\tau \leq t \leq \tau,$$

*and*

$$h_\tau(t) = \begin{cases} D_\tau(t-\tau)a(t-\tau), & 0 \leq t \leq \tau, \\ D_\tau(t+\tau)a(t+\tau), & -\tau \leq t < 0, \end{cases}$$

*respectively. Then for all $\epsilon > 0$ there exists a positive integer $N$ such that*

$$\mathcal{H}_\tau = \mathcal{R}_\tau^{(1)} + \mathcal{E}_\tau^{(1)} \tag{3.1}$$

*and*

$$\mathcal{K}_\tau - \mathcal{A}_\tau = \mathcal{E}_\tau^{(2)} \tag{3.2}$$

*where $\mathcal{R}_\tau^{(1)}$ is a self-adjoint operator with rank $\mathcal{R}_\tau^{(1)} \leq N$ and $\mathcal{E}_\tau^{(i)}(i = 1, 2)$ are self-adjoint operators satisfying $||\mathcal{E}_\tau^{(i)}|| \leq \epsilon$, for $i = 1, 2$.*

**PROOF:** Noting that $\lim_{\tau\to\infty} ||\hat{D}_\tau * \hat{a} - \hat{a}||_\infty = 0$, we can get (3.2) by using the argument in Theorem 1. In addition, as we have

$$\mathcal{H}_\tau = \mathcal{P}_\tau^{(u,0)} - \mathcal{K}_\tau,$$

it follows by using Lemma 6 that (3.1) holds. $\square$

As an immediate corollary, we can show that each $\alpha\mathcal{I} + \mathcal{P}_\tau^{(u,v)}$, $0 \leq v < u$, is also a good approximation for $\alpha\mathcal{I} + \mathcal{A}_\tau$.

**COROLLARY 1** *Let $a(t) \in L_1(-\infty, \infty)$ and its Fourier transform $\hat{a}(t) \geq 0$. Then for all $\epsilon > 0$ and $0 \leq v < u$, there exist a positive integer $N$ and a $\tau^* > 0$ such that for all $\tau > \tau^*$, the spectrum of $(\alpha\mathcal{I} + \mathcal{P}_\tau^{(u,v)})^{-1/2}(\alpha\mathcal{I} + \mathcal{A}_\tau)(\alpha\mathcal{I} + \mathcal{P}_\tau^{(u,v)})^{-1/2}$ has at most $N$ eigenvalues outside $(1 - \epsilon, 1 + \epsilon)$.*

**PROOF:** It is obvious that the function $\tilde{a}(t)$ given by

$$\tilde{a}(t) = \begin{cases} e^{2\pi iv/u} a(t), & 0 \leq t \leq \tau \\ e^{-2\pi iv/u} a(t), & -\tau \leq t \leq 0, \end{cases}$$

is conjugate symmetric in $L_1(-\infty, \infty)$. By (2.7), we have

$$\mathcal{P}_\tau^{(u,v)} = \mathcal{K}_\tau + \tilde{\mathcal{H}}_\tau,$$

where $\tilde{\mathcal{H}}_\tau$ is a self-adjoint convolution operator with its kernel function given by

$$\tilde{h}_\tau(t) = \begin{cases} D_\tau(t - \tau)\tilde{a}(t - \tau), & 0 \leq t \leq \tau, \\ D_\tau(t + \tau)\tilde{a}(t + \tau), & -\tau \leq t < 0. \end{cases}$$

By (3.1) and (3.2) in Lemma 6, it is straightforward to prove that for all $\epsilon > 0$, there exist a positive integer $N$ and a $\tau^* > 0$ such that for all $\tau > \tau^*$, there exists a decomposition

$$\mathcal{A}_\tau - \mathcal{P}_\tau^{(u,v)} = \mathcal{R}_\tau^{(u,v)} + \mathcal{E}_\tau^{(u,v)}$$

with self-adjoint operators $\mathcal{R}_\tau^{(u,v)}$ and $\mathcal{E}_\tau^{(u,v)}$ satisfying rank $\mathcal{R}_\tau^{(u,v)} \leq N$ and $||\mathcal{E}_\tau^{(u,v)}||_2 \leq \epsilon$. As all the eigenvalues of the operator $\alpha\mathcal{I} + \mathcal{P}_\tau^{(u,v)}$ are positive and uniformly bounded (c.f. Lemma 1), The result follows. $\square$

**PROOF OF THEOREM 2:** For $0 \le v < u$, the spectra of both operators

$$(\alpha\mathcal{I} + \mathcal{P}_\tau^{(u,v)})^{-1/2}(\alpha\mathcal{I} + \mathcal{A}_\tau)(\alpha\mathcal{I} + \mathcal{P}_\tau^{(u,v)})^{-1/2}$$

and

$$(\alpha\mathcal{I} + \mathcal{A}_\tau^{(u,v)})^{1/2}(\alpha\mathcal{I} + \mathcal{P}_\tau)^{-1}(\alpha\mathcal{I} + \mathcal{A}_\tau^{(u,v)})^{1/2}$$

are the same. By Corollary 1, we deduce that for $0 \le v < u$, the spectra of $(\alpha\mathcal{I} + \mathcal{A}_\tau)^{\frac{1}{2}}(\alpha\mathcal{I} + \mathcal{P}_\tau^{(u,v)})^{-1}(\alpha\mathcal{I} + \mathcal{A}_\tau)^{\frac{1}{2}}$ is clustered around 1, i.e.

$$(\alpha\mathcal{I} + \mathcal{A}_\tau)^{\frac{1}{2}}(\alpha\mathcal{I} + \mathcal{P}_\tau^{(u,v)})^{-1}(\alpha\mathcal{I} + \mathcal{A}_\tau)^{\frac{1}{2}} = \mathcal{I} + \mathcal{L}_\tau^{(u,v)} + \mathcal{V}_\tau^{(u,v)},$$

where $\mathcal{L}_\tau^{(u,v)}$ is a self-adjoint operator of rank independent of $\tau$ and $\mathcal{V}_\tau^{(u,v)}$ is a self-adjoint operator with norm less than $\epsilon$. We note that by (2.11)

$$(\alpha\mathcal{I} + \mathcal{A}_\tau)^{\frac{1}{2}}\mathcal{B}_\tau^{(u)}(\alpha\mathcal{I} + \mathcal{A}_\tau)^{\frac{1}{2}} = \frac{1}{u}\sum_{v=0}^{u-1}\left(\mathcal{I} + \mathcal{L}_\tau^{(u,v)} + \mathcal{V}_\tau^{(u,v)}\right) = \mathcal{I} + \mathcal{L}_\tau^{(u)} + \mathcal{V}_\tau^{(u)}, \qquad (3.3)$$

where $\mathcal{L}_\tau^{(u)} = \sum_{v=0}^{u-1}\mathcal{L}_\tau^{(u,v)}$ and $\mathcal{V}_\tau^{(u)} = \sum_{v=0}^{u-1}\mathcal{V}_\tau^{(u,v)}$. As $u$ is independent of $\tau$, the rank of $\mathcal{L}_\tau^{(u)}$ is also independent of $\tau$ and $||\mathcal{V}_\tau^{(u)}||_2 \le \epsilon$. Finally, we just note that the operators $(\alpha\mathcal{I} + \mathcal{A}_\tau)^{\frac{1}{2}}\mathcal{T}_\tau^{(u)}(\alpha\mathcal{I} + \mathcal{A}_\tau)^{\frac{1}{2}}$ and $(\mathcal{B}_\tau^{(u)})^{\frac{1}{2}}(\alpha\mathcal{I} + \mathcal{A}_\tau)(\mathcal{B}_\tau^{(u)})^{\frac{1}{2}}$ have same spectra, the theorem follows. $\square$

It follows easily from Theorem 2 that the conjugate gradient method, when applied to solving preconditioned operator equations

$$[\mathcal{B}_\tau^{(u)}(\sigma\mathcal{I} + \mathcal{A}_\tau)x_\tau](t) = (\mathcal{B}_\tau^{(u)}g)(t), \quad 0 \le t \le \tau,$$

converges superlinearly, see Axelsson [2, pp.24-28].

# 4 Numerical Examples

In this section, we use the rectangular rule to discretize the finite section Wiener-Hopf integral operators and $\{\omega_v\}$-circulant integral operators when the kernel functions $a(t)$ are continuous. Numerical integration with rectangular quadrature formula using $n$ points yields $n$-by-$n$ Toeplitz matrices $I_n + A_{\tau,n}$ and $n$-by-$n$ $\{\omega_v\}$-circulant matrices $I_n + P_{\tau,n}^{(u,v)}$ for the finite section Wiener-Hopf integral operators and $\{\omega_v\}$-circulant integral operators respectively. Besides, the following are the properties of the discretization matrices $A_{\tau,n}$ and $P_{\tau,n}^{(u,v)}$.

1. As $a(t)$ is conjugate symmetric, $\alpha I_n + A_{\tau,n}$ is an Hermitian Toeplitz matrix. The matrix-vector multiplications $A_{\tau,n}\mathbf{y}_n$ can be obtained by first embedding $A_{\tau,n}$ into $2n$-by-$2n$ circulant matrices and using $2n$-dimensional fast Fourier transforms (FFTs), see Strang [14].

2. The matrix-vector multiplication $(\alpha I_n + P_{\tau,n}^{(u,v)})^{-1}\mathbf{y_n}$ can be computed in $O(n \log n)$ by using $n$-dimensional FFTs. We let $B_{\tau,n}^{(u)}$ be the sum of the matrices $(1/u)\sum_{v=0}^{u-1}(\alpha I_n + P_{\tau,n}^{(u,v)})^{-1}$. Thus, $B_{\tau,n}^{(u)}$ are the discretization matrices of our preconditioners. We note that the matrix-vector product $B_{\tau,n}^{(u)}\mathbf{y}_n$ can be computed in $O(un \log n)$ operations. When $u$ is chosen to be independent of $\tau$ and $n$, the cost per iteration of the preconditioned conjugate gradient method is $O(n \log n)$ operations.

3. On parallel machines using Single Instruction stream, Multiple Data stream (SIMD) architecture (see for instance Aki [1, p.238]), the real time required by $n$-dimensional FFT is of $O(\log n)$ operations. Therefore, the cost per iteration is reduced to $O(\log n)$ operations.

In the following, we test the performance of our preconditioners in solving Wiener-Hopf equations. Three problems are used and their kernel functions $a(t)$ and the right hand side functions $g(t)$ are

(i)

$$a(t) = e^{-|t|}, \quad \forall t \in \mathbf{R},$$

and

$$g(t) = \begin{cases} e^{-t}(e^{\gamma} - 1), & t > \gamma, \\ \gamma + 2 - e^{-t} - e^{t-\gamma}, & 0 \le t \le \gamma; \end{cases}$$

(ii)

$$a(t) = \begin{cases} 1 - |t|, & -1 \le t \le 1, \\ e^{-|t|}, & |t| > 1, \end{cases}$$

and

$$g(t) = \begin{cases} e^{-t}(e^{\gamma} - 1), & t \ge \gamma + 1, \\ 0.5 - \gamma + e^{-1} - t + \dfrac{(t-\gamma)^2}{2} - e^{-t}, & \gamma \le t < \gamma + 1, \\ \gamma + 0.5 + \gamma + e^{-1} - t - \dfrac{(t-\gamma)^2}{2} - e^{-t}, & \gamma - 1 \le t < \gamma, \\ \gamma + 0.5 + e^{-1} + t - \dfrac{t^2}{2} - e^{t-\gamma}, & 1 \le t < \gamma - 1, \\ \gamma + e^{-1} + 0.5 + t - \dfrac{t^2}{2} - e^{t-\gamma}, & 0 \le t < 1; \end{cases}$$

(iii)

$$a(t) = \begin{cases} \dfrac{1}{|t|^{0.5}}, & -1 \le t \le 1, \\ e^{-|t|}, & |t| > 1, \end{cases}$$

and

$$g(t) = \begin{cases} e^{-t}(e^{\gamma} - 1), & t \ge \gamma + 1, \\ e^{-1} + \dfrac{[1 - (t-\gamma)^{0.5}]}{2} - e^{-t}, & \gamma \le t < \gamma + 1, \\ \gamma + e^{-1} + \dfrac{[1 - (t-\gamma)^{0.5}]}{2} - e^{-t}, & \gamma - 1 \le t < \gamma, \\ \gamma + 2e^{-1} + 4 - e^{t-\gamma} - e^{-t}, & 1 \le t < \gamma - 1, \\ \gamma + e^{-1} + 4 - e^{t-\gamma}, & 0 \le t < 1. \end{cases}$$

For these $a(t)$ and $g(t)$, the solutions of their corresponding Wiener-Hopf equations are all

$$x(t) = \begin{cases} 1, & 0 \le t \le \gamma, \\ 0, & t > \gamma. \end{cases}$$

The discretization matrices $\alpha I_n + A_{\tau,n}$ and $\alpha I_n + P_{\tau,n}^{(u,v)}$ of $\alpha\mathcal{I} + \mathcal{A}_\tau$ and $\alpha\mathcal{I} + \mathcal{P}_\tau^{(u,v)}$ are formed respectively by using the rectangular rule on these testing kernel functions. We note that the kernel functions in (ii) and (iii) has a jump $t = 1$, we use a simple average as the values of $a(1)$. Moreover, the kernel function $a(t)$ in (iii) is undefined at $t = 0$ (i.e. $a(t)$ has at singularity at $t = 0$). In this case, we just replace the value of $a(0)$ by zero. Moreover, $\gamma$ is set to be 8 in the following numerical tests.

In the tests, random vectors are used to be our initial guesses. The stopping criterion is $||\mathbf{r}_k||_2/||\mathbf{r}_0||_2 < 10^{-7}$, where $\mathbf{r}_k$ is the residual vector of preconditioned conjugate gradient method after $k$ iterations. The parameters $\alpha$ are chosen such that the discretization matrices of the finite section Wiener-Hopf integral operators are positive definite. All computations are done on Matlab. Tables 1a-3b show the numbers of iterations required for convergence with different choices of preconditioners. In the tables, $\mathcal{I}$ denotes no preconditioner is used, $\mathcal{B}_\tau^{(u)}$ are the proposed preconditioners. We remark that $\mathcal{B}_\tau^{(1)}$ are in fact the circulant integral operators. As for the comparison, the "optimal" circulant integral operators proposed by Gohberg *et. al.* [9]. are also used in our numerical tests. We denote it by $\mathcal{G}_\tau$.

We see in the tables that when $\tau$ is fixed, as $n$ increases, the number of iterations of the preconditioned systems are almost kept constant while that of the non-preconditioned systems increases. When the factor $\tau/n$ is fixed as $\tau$ increases, we see that the number of iterations of the preconditioned systems are almost kept constant and less than that of non-preconditioned systems. This observation is consistent with our theoretical result. We recall in Theorem 2 that the spectra of the preconditioned systems are clustered around 1 and then it leads the fast convergence rate of the method. We also note that the preconditioners $\mathcal{B}_\tau^{(2)}$ performs better than the other preconditioners using circulant integral operators. Finally, we observe in all tests that the discrete $L_2[0,\tau]$ norm error

$$\frac{1}{n}\sum_{j=1}^{n-1}\left|x_\tau(\frac{j\tau}{n}) - x_\tau^{(k)}(\frac{j\tau}{n})\right|^2$$

decreases like $O(1/n)$ where $x_\tau^{(k)}$ is $k$th iterant of the preconditioned conjugate gradient method.

| | $\tau$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 16 | | | | 32 | | | | 64 | | | |
| $n$ | $I$ | $\mathcal{B}_\tau^{(2)}$ | $\mathcal{B}_\tau^{(1)}$ | $\mathcal{G}_\tau$ | $I$ | $\mathcal{B}_\tau^{(2)}$ | $\mathcal{B}_\tau^{(1)}$ | $\mathcal{G}_\tau$ | $I$ | $\mathcal{B}_\tau^{(2)}$ | $\mathcal{B}_\tau^{(1)}$ | $\mathcal{G}_\tau$ |
| 16 | 12 | 2 | 3 | 6 | ** | ** | ** | ** | ** | ** | ** | ** |
| 32 | 17 | 2 | 3 | 6 | 14 | 2 | 3 | 6 | ** | ** | ** | ** |
| 64 | 26 | 2 | 3 | 6 | 23 | 2 | 3 | 6 | 14 | 2 | 3 | 5 |
| 128 | 35 | 2 | 3 | 7 | 33 | 2 | 3 | 6 | 25 | 2 | 3 | 5 |
| 256 | 40 | 2 | 3 | 6 | 46 | 2 | 3 | 6 | 40 | 2 | 3 | 6 |
| 512 | 43 | 2 | 3 | 6 | 53 | 2 | 3 | 6 | 55 | 2 | 3 | 5 |
| 1024 | 43 | 2 | 3 | 6 | 55 | 2 | 3 | 6 | 64 | 2 | 3 | 5 |
| 2048 | 43 | 2 | 3 | 6 | 57 | 2 | 3 | 6 | 68 | 2 | 3 | 5 |

Table 1a. Number of Iterations for (i) with $\alpha = 0.01$

| | $\tau$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 128 | | | | 256 | | | | 512 | | | |
| $n$ | $I$ | $\mathcal{B}_\tau^{(2)}$ | $\mathcal{B}_\tau^{(1)}$ | $\mathcal{G}_\tau$ | $I$ | $\mathcal{B}_\tau^{(2)}$ | $\mathcal{B}_\tau^{(1)}$ | $\mathcal{G}_\tau$ | $I$ | $\mathcal{B}_\tau^{(2)}$ | $\mathcal{B}_\tau^{(1)}$ | $\mathcal{G}_\tau$ |
| 16 | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** |
| 32 | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** |
| 64 | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** |
| 128 | 14 | 2 | 3 | 5 | ** | ** | ** | ** | ** | ** | ** | ** |
| 256 | 25 | 2 | 3 | 5 | 14 | 2 | 3 | 5 | ** | ** | ** | ** |
| 512 | 43 | 2 | 3 | 5 | 25 | 2 | 3 | 5 | 14 | 2 | 3 | 4 |
| 1024 | 61 | 2 | 3 | 5 | 43 | 2 | 3 | 5 | 25 | 2 | 3 | 5 |
| 2048 | 72 | 2 | 3 | 5 | 62 | 2 | 3 | 5 | 43 | 2 | 3 | 5 |

Table 1b. Number of Iterations for (i) with $\alpha = 0.01$

| | $\tau$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 16 | | | | 32 | | | | 64 | | | |
| $n$ | $I$ | $\mathcal{B}_\tau^{(2)}$ | $\mathcal{B}_\tau^{(1)}$ | $\mathcal{G}_\tau$ | $I$ | $\mathcal{B}_\tau^{(2)}$ | $\mathcal{B}_\tau^{(1)}$ | $\mathcal{G}_\tau$ | $I$ | $\mathcal{B}_\tau^{(2)}$ | $\mathcal{B}_\tau^{(1)}$ | $\mathcal{G}_\tau$ |
| 16 | 8 | 3 | 5 | 6 | ** | ** | ** | ** | ** | ** | ** | ** |
| 32 | 19 | 4 | 7 | 7 | 10 | 3 | 5 | 6 | ** | ** | ** | ** |
| 64 | 27 | 5 | 8 | 9 | 23 | 4 | 6 | 7 | 10 | 3 | 5 | 5 |
| 128 | 35 | 6 | 10 | 10 | 33 | 5 | 9 | 9 | 26 | 4 | 7 | 7 |
| 256 | 38 | 7 | 11 | 10 | 42 | 6 | 10 | 10 | 38 | 5 | 9 | 9 |
| 512 | 38 | 7 | 11 | 11 | 48 | 6 | 11 | 10 | 48 | 5 | 10 | 10 |
| 1024 | 39 | 7 | 11 | 11 | 48 | 6 | 11 | 11 | 53 | 6 | 10 | 10 |
| 2048 | 38 | 7 | 11 | 11 | 47 | 6 | 11 | 11 | 53 | 6 | 10 | 11 |

Table 2a. Number of Iterations for (ii) with $\alpha = 0.08$

| | $\tau$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 128 | | | | 256 | | | | 512 | | | |
| $n$ | $I$ | $\mathcal{B}_\tau^{(2)}$ | $\mathcal{B}_\tau^{(1)}$ | $\mathcal{G}_\tau$ | $I$ | $\mathcal{B}_\tau^{(2)}$ | $\mathcal{B}_\tau^{(1)}$ | $\mathcal{G}_\tau$ | $I$ | $\mathcal{B}_\tau^{(2)}$ | $\mathcal{B}_\tau^{(1)}$ | $\mathcal{G}_\tau$ |
| 16 | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** |
| 32 | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** |
| 64 | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** |
| 128 | 10 | 3 | 5 | 5 | ** | ** | ** | ** | ** | ** | ** | ** |
| 256 | 27 | 4 | 7 | 7 | 10 | 3 | 5 | 5 | ** | ** | ** | ** |
| 512 | 40 | 5 | 8 | 8 | 27 | 4 | 7 | 7 | 10 | 3 | 5 | 5 |
| 1024 | 53 | 5 | 9 | 10 | 40 | 5 | 8 | 8 | 27 | 4 | 6 | 7 |
| 2048 | 60 | 6 | 10 | 10 | 54 | 5 | 10 | 9 | 40 | 5 | 8 | 8 |

Table 2b. Number of Iterations for (ii) with $\alpha = 0.08$

| | $\tau$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 16 | | | | 32 | | | | 64 | | | |
| $n$ | $I$ | $\mathcal{B}_\tau^{(2)}$ | $\mathcal{B}_\tau^{(1)}$ | $\mathcal{G}_\tau$ | $I$ | $\mathcal{B}_\tau^{(2)}$ | $\mathcal{B}_\tau^{(1)}$ | $\mathcal{G}_\tau$ | $I$ | $\mathcal{B}_\tau^{(2)}$ | $\mathcal{B}_\tau^{(1)}$ | $\mathcal{G}_\tau$ |
| 16 | 16 | 5 | 5 | 11 | ** | ** | ** | ** | ** | ** | ** | ** |
| 32 | 31 | 5 | 7 | 9 | 35 | 4 | 5 | 12 | ** | ** | ** | ** |
| 64 | 21 | 4 | 7 | 7 | 46 | 4 | 7 | 8 | 76 | 5 | 5 | 12 |
| 128 | 19 | 4 | 7 | 7 | 24 | 4 | 7 | 7 | 51 | 4 | 7 | 8 |
| 256 | 18 | 4 | 6 | 7 | 21 | 4 | 6 | 7 | 25 | 4 | 6 | 7 |
| 512 | 18 | 4 | 6 | 6 | 20 | 4 | 6 | 6 | 21 | 4 | 6 | 6 |
| 1024 | 17 | 4 | 6 | 6 | 19 | 4 | 6 | 6 | 20 | 4 | 6 | 6 |
| 2048 | 19 | 4 | 6 | 6 | 18 | 3 | 6 | 6 | 19 | 3 | 6 | 6 |

Table 3a. Number of Iterations for (iii) with $\alpha = 1.0$

|  | $\tau$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
|  | 128 | | | | 256 | | | | 512 | | | |
| $n$ | $I$ | $\mathcal{B}_\tau^{(2)}$ | $\mathcal{B}_\tau^{(1)}$ | $\mathcal{G}_\tau$ | $I$ | $\mathcal{B}_\tau^{(2)}$ | $\mathcal{B}_\tau^{(1)}$ | $\mathcal{G}_\tau$ | $I$ | $\mathcal{B}_\tau^{(2)}$ | $\mathcal{B}_\tau^{(1)}$ | $\mathcal{G}_\tau$ |
| 16 | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** |
| 32 | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** |
| 64 | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** | ** |
| 128 | 161 | 4 | 5 | 12 | ** | ** | ** | ** | ** | ** | ** | ** |
| 256 | 51 | 4 | 7 | 7 | 332 | 5 | 5 | 12 | ** | ** | ** | ** |
| 512 | 25 | 4 | 7 | 7 | 50 | 4 | 7 | 7 | 678 | 5 | 5 | 13 |
| 1024 | 21 | 4 | 6 | 6 | 24 | 4 | 6 | 7 | 49 | 4 | 7 | 7 |
| 2048 | 19 | 3 | 6 | 6 | 21 | 3 | 6 | 6 | 24 | 4 | 6 | 6 |

Table 3b. Number of Iterations for (iii) with $\alpha = 1.0$

# 5 Concluding Remarks

In this paper, the Dirichlet kernel $\hat{D}_\tau$ is used in constructing the $\{\omega\}$-circulant integral operators $\mathcal{P}_\tau^{(u,v)}$, see Theorem 1. We remark that we can use other kernel functions $\hat{W}_\tau$ instead as long as $\hat{W}_\tau * \hat{a}$ converges to $\hat{a}$ uniformly, for example the Fejér function. We just replace $D_\tau$ by $W_\tau$ in (2.7). Then (2.9) in Lemma 3 holds and therefore the spectra of these preconditioned operators are clustered around 1 and the preconditioned conjugate gradient method converges sufficiently fast.

# References


[1] S. Aki, *The Design and Analysis of Parallel Algorithms*, Prentice-Hall International Inc., London, 1989.

[2] O. Axelsson and V. Barker, *Finite Element Solution of Boundary Value Problems, Theory and Computation*, Academic Press, New York, 1984.

[3] R. Chan, X. Jin and M. Ng, *Circulant Integral Operators as Preconditioners for Wiener-Hopf Equations*, To appear in Operator Theory and Integral Equations.

[4] R. Chan and M. Ng, *Toeplitz Preconditioners for Hermitian Toeplitz Systems*, Lin. Alg. and Appl., V190 (1993), pp.181–208.

[5] P. Davis, *Circulant Matrices*, John Wiley & Sons, Inc., New York, 1979.

[6] M. Dryja and O. Widlund, *Some Domain Decomposition Algorithms for Elliptic Problems*, Proceedings of the Conference on Iterative Methods for Large Linear Systems, Austin, Texas, October, 1989, Academic Press.

[7] I. Gohberg and I. Feld́man, *Convolution Equations and Projection Methods for Their Solution*, Transl. Math. Monographs, V41 (1974), Amer. Math. Soc. Providence, RI.

[8] I. Gohberg and S. Goldberg, *Basic Operator Theory*, Birkhäuser, Boston, Basel, Stuttgart, 1981.

[9] I. Gohberg, M. Hanke and I. Koltracht, *Fast Preconditioned Conjugate Gradient Algorithms for Wiener-Hopf Integral Equations*, To appear in SIAM J. Numer. Anal.

[10] C. Green, *Integral Equation Method*, London, Nelson, 1969.

[11] U. Grenander and G. Szegö, *Toeplitz Forms and Their Applications*, 2nd. ed., Chelsea, New York, 1984.

[12] H. Hochstadt, *Integral Equations*, John Wiley & Sons, Inc., New York, 1989.

[13] R. Kress, *Linear Integral Equations*, Springer-Verlag, New York, 1980.

[14] G. Strang, *A Proposal for Toeplitz Matrix Calculations*, Stud. Appl. MAth., V74 (1986), pp.171-176.

[15] P. Zabreyko, A. Koshelev, M. Krasnosel'skii, S. Mikhlin, L. Rakovshchik and V. Ya, *Integral Equations – a Reference Text*, Noordhoff International Publishing, Leyden, 1975.