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Two-Grid Methods for Banded Linear Systems from DCT III Algebra

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SUMMARY

We describe a two-grid and a multigrid method for linear systems whose coefficient matrices are point or block matrices from the cosine algebra generated by a polynomial. We show that the convergence rate of the two-grid method is constant independent of the size of the given matrix. Numerical examples from differential and integral equations are given to illustrate the convergence of both the two-grid and the multigrid method. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: DCT-III matrix algebra; band matrices; two-grid method; multigrid method

1. INTRODUCTION

Consider solving a linear system $A_n\mathbf{x}=\mathbf{b}$ where $A_n \in \mathbb{R}^{n\times n}$. Given an iterative method

$$\mathbf{x}^{(j+1)} = V_n\mathbf{x}^{(j)} + \omega\mathbf{b}, \tag{1}$$

where $V_n := I_n - \omega A_n$ and a full-rank matrix $p_n^k \in \mathbb{R}^{n\times k}$ with $k \le n$, a Two-Grid Method (TGM) for solving $A_n\mathbf{x}=\mathbf{b}$ is defined by the global iteration matrix:

$$TGM(V_n, p_n^k) = V_n\left[I_n - p_n^k\left((p_n^k)^t A_n p_n^k\right)^{-1}(p_n^k)^t A_n\right],$$

see [5]. The TGM becomes a multigrid method (MGM) if the coarse-grid system with the coefficient matrix $(p_n^k)^t A_n p_n^k$ is solved recursively by using the TGM scheme.

In this paper, we consider A_n that are point or block matrices in the DCT III algebra generated by a polynomial. This kind of matrices appears in the solution of differential

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equations and integral equations, see [3, 6, 9] for instance. We will show that the convergence rate of the TGM is constant independent of n, i.e. $\|TGM(V_n, p_n^k)\| < 1$. It is the first step in establishing the optimality of the MGM (see [7]). In fact, the total cost of MGM will be of $O(n)$ operations if one can find projection operators p_n^k such that

[**a**] the matrix vector product involving p_n^k costs $O(n)$ operations;

[**b**] the coarse grid matrix $A_k = (p_n^k)^t A_n p_n^k$ is also a matrix in the DCT III algebra generated by a polynomial, and can be formed within $O(n)$ operations; and

[**c**] the convergence rate of the MGM is independent of n.

The conditions [**a**]–[**c**] imply that the MGM will be optimal in the sense of Axelsson and Neytcheva [2], i.e. the problem of solving a linear system with coefficient matrix A_n is asymptotically of the same cost as the direct problem of multiplying A_n to a vector.

The paper is organized as follows. In §2, we define our TGM that satisfies conditions [**a**] and [**b**]. In §3, we show that $\|TGM(V_n, p_n^k)\| < 1$. Finally, §4 gives some numerical results and §5 provides some open problems and future work.

2. TWO-GRID METHOD FOR DCT III MATRICES

In this section we describe our TGM for DCT III matrices that satisfies requirements [**a**] and [**b**]. We start with the unilevel case (i.e. point matrices). Let f be a real-valued even trigonometric polynomial of degree c defined over $(0, 2\pi]$. The DCT III matrix of order n generated by f is defined as $S_n(f) = Q_n \Delta_n(f) Q_n^t$, where $\Delta_n(f) = \mathrm{diag}_{0 \le j \le n-1} f(\pi j/n)$ and

$$Q_n = \left[\sqrt{\frac{2-\delta_{j,1}}{n}} \cos\left\{ \frac{(i-1)(2j-1)\pi}{2n} \right\} \right]_{i,j=1}^{n}, \quad \delta_{1,1} = 1, \delta_{j,1} = 0 \ \text{ if } j \neq 1, \tag{2}$$

is the DCT III transform matrix. Let $A_n = S_n(f)$. Thus A_n is a symmetric band matrix of bandwidth $2c+1$.

In order that the coarse grid matrix A_k ($k = n/2$) is a DCT III matrix, we let $p_n^k := S_n(p) T_n^k$ where p is an even trigonometric polynomial to be defined later, and $T_n^k \in \mathbb{R}^{n \times k}$ is given by

$$\left[T_n^k\right]_{i,j} = \begin{cases} 1 & \text{for } \ i \in \{2j-1, 2j\}, \ j = 1, \dots, k, \\ 0 & \text{otherwise.} \end{cases}$$

The operator T_n^k represents a spectral link between the space of the frequencies of size n and the corresponding space of frequencies of size k according to the following Lemma.

Lemma 1. *Let Q_m be given as in (2) with size m. Then*

$$(T_n^k)^t Q_n = Q_k [\Phi_k, \Theta_k \Pi_k], \tag{3}$$

where

$$\Phi_k = \mathrm{diag}_{s=1,\dots,k} \left[\sqrt{2} \cos\left(\frac{(s-1)\pi}{4k} \right) \right], \quad \Theta_k = \mathrm{diag}_{s=1,\dots,k} \left[-\sqrt{2} \sin\left(\frac{(s-1)\pi}{4k} \right) \right],$$

and Π_k is the permutation matrix $(1, 2, \dots, k) \mapsto (1, k, k-2, \dots, 2)$.

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Proof: We break (3) into three different cases:

$$[Q_n]_{2s-1,t} + [Q_n]_{2s,t} = [\Phi_k]_{t,t}[Q_k]_{s,t}, \quad 1 \le s, t \le k, \tag{4}$$

$$[Q_n]_{2s-1,k+1} + [Q_n]_{2s,k+1} = 0, \quad 1 \le s \le k,$$

and $[Q_n]_{2s-1,t+k} + [Q_n]_{2s,t+k} = [\Theta_k]_{k+2-t,k+2-t}[Q_k]_{s,k+2-t}, \quad 1 \le s \le k, 2 \le t \le k.$

The first two identities can be proven using (2) and the cosine summation formula. For the last one, we can simply observe that (4) holds even for $t > k$, and that $[\Phi_k]_{t+k,t+k} = [\Theta_k]_{k+2-t,k+2-t}$ and $[Q_k]_{s,t+k} + [Q_k]_{s,k+2-t} = 0$. $\square$

Notice that $A_k = (p_n^k)^t S_n(f) p_n^k = (T_n^k)^t S_n(p) S_n(f) S_n(p) T_n^k = (T_n^k)^t Q_n \Delta_n(fp^2) Q_n^t T_n^k$. Hence by Lemma 1, it is straightforward to verify that $A_k = S_k(\hat{f})$ where

$$\hat{f}(x) = 2[\cos^2(x/4) f(x/2) p^2(x/2) + \sin^2(x/4) f(\pi - x/2) p^2(\pi - x/2)], \quad x \in [0, \pi]. \tag{5}$$

From this, it is easy to obtain the Fourier coefficients of $\hat{f}(x)$ and hence the nonzero entries of A_k.

To ensure the convergence of our TGM, we choose p as follows: if f has a unique zero $x^0 \in [0, \pi]$, then

$$p(x) = [2 - 2\cos(x - (\pi - x^0))]^{\lceil \beta/2 \rceil} \sim |x - (\pi - x^0)|^{2\lceil \beta/2 \rceil}, \quad x \in [0, \pi], \tag{6}$$

$$\beta \ \ge \ \beta_{\min} = \min\left\{ i \ \middle| \ \lim_{x \to x^0} \frac{\sin^2(x/2)}{\cos^2(x/2)} \frac{|x - x^0|^{2i}}{f(x)} < \infty \right\}, \tag{7}$$

$$0 \ < \ p^2(x) + p^2(\pi - x). \tag{8}$$

Here the equivalence relation $f \sim g$ indicates that f and g are both nonnegative and there exist positive constants c and C such that $cg(x) \le f(x) \le Cg(x)$, uniformly with respect to x.

If f has more than one zero in $[0, \pi]$, then p will be the product of the polynomials of the kind (6), satisfying the condition (7) for every single zero and globally the condition (8). We point out that conditions (6) and (8) are not surprising and indeed are in common with the conditions on other structures such as τ, symmetric Toeplitz and circulant matrices (see e.g. [8, 10]). Condition (7) is new and is special for the DCT III algebra.

We note that if f is a trigonometric polynomial of degree c, then f can have a zero of order at most $2c$. If none of the root of f are at π, then by (6) and (7), $\beta \le c$. Therefore the degree of p is less than or equal to $\lceil c/2 \rceil$. If π is one of the roots of f, then $\beta = \min\{i : \lim_{x \to \pi} (x - \pi)^{2i} / (\cos^2(x/2) f(x)) < +\infty\}$. In this case, $\beta \le c+1$ and consequently the degree of p is less than or equal to $\lceil (c+1)/2 \rceil$. In both cases, $S_n(p)$ is banded, and hence the cost of a matrix vector product involving p_n^k is $O(n)$. Thus the condition [**a**] in §1 is satisfied. Regarding the condition [**b**], the representation of A_k can be obtained formally in $O(1)$ operations since $A_k = S_k(\hat{f})$ and $\hat{f}$ is given in (5). In fact, if $0 \le x^0 \le \pi/2$ is a zero of f, then by (6), $p(\pi - x^0) = 0$ and hence by (5), $\hat{f}(2x^0) = 0$, i.e. $y^0 = 2x^0$ is a zero of $\hat{f}$. Furthermore, because $p(\pi - x^0) = 0$, by (8), $p(x^0) > 0$ and hence the orders of x^0 and y^0 are the same. Similarly, we can show that if $\pi/2 \le x^0 < \pi$, then $y^0 = 2(\pi - x^0)$ is a root of $\hat{f}$ with the same order as x^0; and if $x^0 = \pi$, then $y^0 = 0$ with order equals to the order of x^0 plus two. More precisely, we can obtain the roots of $\hat{f}$ and their orders by knowing the roots

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of f and their orders. Finally, each iteration of (1) costs $O(n)$ operations as A_n is banded. In conclusion, each iteration of our TGM requires $O(n)$ operations.

For 2-level (i.e. block) banded DCT III matrices, i.e. matrices of the form

$$(Q_n \otimes Q_n)\Delta_{n^2}(f(x_1,x_2))(Q_n \otimes Q_n)^t$$

for some 2-variate trigonometric polynomial f, we can extend our TGM directly to them by using tensor arguments. In particular, we choose $p(x_1,x_2)$ to be an even trigonometric polynomial that satisfies the following conditions: if $f(x_1,x_2)$ has a unique zero $(x_1^0,x_2^0) \in [0,\pi]^2$, then

$$p(x_1,x_2) \sim \prod_{(\hat{x}_1^0,\hat{x}_2^0)\in M(x_1^0,x_2^0)} \left(\sum_{r=1}^{2} |x_r - \hat{x}_r^0|^{2\lceil \beta/2 \rceil} \right),$$

$$\begin{aligned} \beta &\geq \beta_{\min} = \min\left\{ i \left| \sum_{r=1}^{2} \lim_{x_r \to x_r^0} \frac{\sin^2(x_r/2)}{\cos^2(x_r/2)} \frac{|x_r - x_r^0|^{2i}}{f(x_1,x_2)} < \infty \right. \right\}, \\ 0 &< \sum_{(\hat{x}_1,\hat{x}_2)\in M(x_1,x_2)\cup\{(x_1,x_2)\}} p^2(\hat{x}_1,\hat{x}_2) \end{aligned}$$

with $M(x_1,x_2) = \{(x_1,\pi - x_2),(\pi - x_1,x_2),(\pi - x_1,\pi - x_2)\}$.

3. PROOF OF CONVERGENCE

In this section, we show the optimality of our TGM by proving that $\|TGM(V_n,p_n^k)\| < 1$. This is a weaker version of condition [**c**] in §1. We give the proof for the point case, as the proof for the 2-level case is similar by tensor argument. In the following, $\|\cdot\|_2$ denotes the Euclidean norm, and $\|\cdot\|_X = \|X^{1/2}\cdot\|_2$ for nonnegative definite X. We start with the following theorem.

Theorem 1. [7] *Let A_n be positive definite and V_n be defined as in (1). Suppose that there exist positive α and γ independent of n such that*

$$\|V_n\mathbf{x}_n\|_{A_n}^2 \leq \|\mathbf{x}_n\|_{A_n}^2 - \alpha\|\mathbf{x}_n\|_{A_n\mathrm{diag}(A_n)^{-1}A_n}^2, \quad \forall \mathbf{x}_n \in \mathbb{R}^n, \tag{9}$$

and

$$\min_{\mathbf{y}_k\in\mathbb{R}^k} \|\mathbf{x}_n - p_n^k\mathbf{y}_k\|_{\mathrm{diag}(A_n)}^2 \leq \gamma\|\mathbf{x}_n\|_{A_n}^2, \quad \forall \mathbf{x}_n \in \mathbb{R}^n. \tag{10}$$

Then, $\gamma \geq \alpha$ and $\|TGM(V_n,p_n^k)\|_{A_n} \leq \sqrt{1-\alpha/\gamma} < 1$.

Thus our TGM will converge linearly if we can verify (9) and (10). We first verify (9).

Lemma 2. *Let $A_n = S_n(f)$ with f being a nonnegative trigonometric polynomial and let $V_n = I_n - A_n/\|f\|_\infty$ in (1). Then (9) holds.*

Proof: Since $A_n = S_n(f)$, we can check that

$$[A_n]_{ii} = \sum_{j=1}^{n} f\left(\frac{\pi j}{n}\right)[Q_n]_{i,j}^2 > 0, \qquad 1 \leq i \leq n. \tag{11}$$

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Hence $\hat{a} = \min_i\{[A_n]_{ii}\} > 0$ and is independent of n. Now choose $0 < \alpha \leq \hat{a}/\|f\|_\infty$, then $(I_n - A_n/\|f\|_\infty)^2 \leq I_n - (\alpha/\hat{a})A_n$. Since $\mathrm{diag}(A_n)^{-1} \leq \hat{a}^{-1}I_n$, we have $(I_n - A_n/\|f\|_\infty)^2 \leq I_n - \alpha A_n^{1/2}\mathrm{diag}(A_n)^{-1}A_n^{1/2}$, which is equivalent to (9). $\square$

It remains to verify (10) for which we use a proof technique developed in [8].

Lemma 3. *Let $A_n = S_n(f)$ with f being a nonnegative trigonometric polynomial. Then (10) holds for our TGM.*

Proof: Let $\tilde{a} = \max_i\{[A_n]_{ii}\}$. By (11), $0 < \tilde{a} < \infty$. Notice that $\|\cdot\|^2_{\mathrm{diag}(A_n)} \leq \tilde{a}\|\cdot\|_2^2$. Hence it is enough to prove that there exists $\gamma > 0$ such that

$$\min_{\mathbf{y}_k \in \mathbb{R}^k} \|\mathbf{x}_n - p_n^k \mathbf{y}_k\|_2^2 \leq (\gamma/\tilde{a})\|\mathbf{x}_n\|^2_{A_n}, \quad \forall \mathbf{x}_n \in \mathbb{R}^n.$$

We will establish this by showing that there exists a positive γ independent of n such that

$$\|\mathbf{x}_n - p_n^k\{\left[(p_n^k)^t p_n^k\right]^{-1}(p_n^k)^t\mathbf{x}_n\}\|_2^2 \leq (\gamma/\tilde{a})\|\mathbf{x}_n\|^2_{A_n}, \quad \forall \mathbf{x}_n \in \mathbb{R}^n.$$

But this is equivalent to showing that

$$W_n(p)^t W_n(p) \leq (\gamma/\tilde{a})S_n(f)$$

with $W_n(p) := I_n - p_n^k\left[(p_n^k)^t p_n^k\right]^{-1}(p_n^k)^t$. Here $X \leq Y$ means that $Y - X$ is positive definite. Since $W_n(p)^t W_n(p) = W_n^2(p) = W_n(p)$, the preceding inequality becomes

$$W_n(p) \leq (\gamma/\tilde{a})S_n(f). \tag{12}$$

Recall that $p_n^k = S_n(p)T_n^k$. Using (3), we can permute $Q_n^t W_n(p) Q_n$ into a 2×2 block diagonal matrix where the first block of this matrix is given by

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and the jth block, $j = 2, \ldots, k$, is given by

$$I_2 - \frac{1}{c_j^2 + s_j^2}\begin{bmatrix} c_j^2 & c_j s_j \\ c_j s_j & s_j^2 \end{bmatrix}.$$

Here $c_j = \sqrt{2}\cos(x_j^{[n]}/2)p(x_j^{[n]})$, $s_j = -\sqrt{2}\sin(x_j^{[n]}/2)p(\pi - x_j^{[n]})$, and $x_j^{[n]} = (j-1)\pi/n$.

Due to the continuity of f and p, (12) is proven if

$$I_2 - \frac{1}{\|p[x]\|_2^2}p[x]p^t[x] \leq (\gamma/\tilde{a})\mathrm{diag}(f[x]) \tag{13}$$

where $p[x] = [\cos(x/2)p(x), -\sin(x/2)p(\pi - x)]^t$ and $f[x] = [f(x), f(\pi - x)]^t$. Let

$$R(x) = \mathrm{diag}^{-\frac{1}{2}}(f[x])\left(I_2 - \frac{1}{\|p[x]\|_2^2}p[x]p^t[x]\right)\mathrm{diag}^{-\frac{1}{2}}(f[x])$$

If we can show that all the entries of $R(x)$ are uniformly bounded, then $R(x) \leq (\gamma/\tilde{a})I_2$ for some $\gamma > 0$. Hence by the Sylvester inertia, (13) holds, and we are done.

For $i \neq j$, it holds

$$[R(x)]_{ij} = -\frac{\cos(x/2)\sin(x/2)}{\cos^2(x/2)p^2(x) + \sin^2(x/2)p^2(\pi - x)}\frac{p(x)p(\pi - x)}{\sqrt{f(x)f(\pi - x)}}$$

and we have to consider the following three cases:

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1. $x^0 \in (0,\pi)$: (7) implies that $p(\pi - x)/\sqrt{f(x)} \in L^\infty$, and, up to the change of variables $y = \pi - x$, that $p(y)/\sqrt{f(\pi - y)} \in L^\infty$. In addition, (8) implies that $\cos(x/2)\sin(x/2)/(\cos^2(x/2)p^2(x) + \sin^2(x/2)p^2(\pi - x)) \in L^\infty$.
2. $x^0 = 0$: (7) implies that $\sin(x/2)p(\pi - x)/\sqrt{f(x)} \in L^\infty$, and, up to the change of variables $y = \pi - x$, that $\cos(y/2)p(y)/\sqrt{f(\pi - y)} \in L^\infty$. In addition, (8) implies that $1/(\cos^2(x/2)p^2(x) + \sin^2(x/2)p^2(\pi - x)) \in L^\infty$.
3. $x^0 = \pi$: (7) implies that $p(\pi - x)/(\cos(x/2)\sqrt{f(x)}) \in L^\infty$, and, up to the change of variables $y = \pi - x$, that $p(y)/(\sin(y/2)\sqrt{f(\pi - y)}) \in L^\infty$. In addition (8) implies that $p^2(x)/\sin^2(x/2) + p^2(\pi - x)/\cos^2(x/2) > p^2(x) + p^2(\pi - x) > 0$.

Thus in all cases, $[R(x)]_{ij} \in L^\infty$. The cases $[R(x)]_{jj}$, $j = 1, 2$, can be dealt with in the same manner. $\square$

4. NUMERICAL EXPERIMENTS

We test our TGM and MGM (standard V-cycle) for two types of DCT-III systems with generating functions having zero at 0 (differential like problems) and at π (integral like problems). The projectors are chosen as in §2 and we use the smoothing (1) twice in each iteration with $\omega = \|f\|^{-1}$ and $\omega = 2\|f\|^{-1}$ respectively. In the V-cycle, the exact solution of the system is found by a direct solver when the coarse grid dimension equals to 16^d where $d = 1$ for the point case and $d = 2$ for the block case. In all tables, $\mathbf{x}_e$ denotes the exact solution, and we give the numbers of iterations required for convergence for both the TGM and the MGM. We point out that the CPU timings are consistent with the iteration counts. We stop the iterations when the two norm of the residual is less than 10^{-7}. Finally, we stress that the matrices at every level (except for the coarsest) are never formed since we need only to store the nonzero Fourier coefficients of the symbols at every level for matrix-vector multiplications. Thus, besides the $O(n)$ operations complexity of our V-cycle, the memory requirements is also very low since there are only $O(1)$ nonzero Fourier coefficients of the symbols at every level.

4.1. *Case* $x^0 = 0$

Consider $A_n = S_n(f_q)$ where $f_q(x) = [2 - 2\cos(x)]^q$, which has a unique zero at $x^0 = 0$ of order $2q$. Since $S_n(f_q)$ is singular, here, we consider the solution of the perturbed system $[S_n(f_q) + \frac{f(\pi/n)}{n} \cdot \mathbf{e}\mathbf{e}^t]\mathbf{x} = \mathbf{b}$, where $\mathbf{e}$ is the vector of all ones. Notice that the position of the zero at the coarser levels is exactly the same as at the finest level. Consequently the function $p(x)$ in the projectors is the same at all the subsequent levels. To test condition (7), we tried $p(x) = [2 - 2\cos(\pi - x)]^w$ for different w. According to (7), we must choose w at least equal to 1 if $q = 1$ and at least equal to 2 if $q = 2, 3$. These are confirmed in Table I. We also see that our choice of w guarantees the linear convergence of the multigrid method with rate independent of the size n.

By using tensor arguments, our results plainly extend to the block case. We test it with $f_q(x, y) = f_q(x) + f_q(y)$. The results are shown in Table II. We note that such kind of linear systems arises in the uniform finite difference discretization of elliptic constant coefficient differential equations on a square with Neumann boundary conditions, see e.g [9].

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Table I. Twogrid/Multigrid - $1D$ Case: $(\mathbf{x}_e)_i = i/N$.

	$q=1$		$q=2$		$q=3$		
n	$w=0$	$w=1$	$w=1$	$w=2$	$w=1$	$w=2$	$w=3$
16	25/1	7/1	15/1	13/1	-/1	34/1	32/1
32	26/26	7/7	16/16	15/15	36/36	35/34	34/32
64	27/60	7/7	16/17	16/16	36/63	35/35	35/34
128	28/125	7/7	16/18	16/16	36/123	35/35	35/35
256	28/251	7/7	16/18	16/16	36/225	35/35	35/35
512	29/497	7/7	16/18	16/16	36/391	35/35	35/35

Table II. Twogrid/Multigrid - $2D$ case: $(\mathbf{x}_e)_i = \lfloor i/n_1 \rfloor /n_2 + i \pmod{n_1}/n_1$.

	$q=1$		$q=2$		$q=3$		
$N = n_1 n_2$	$w=0$	$w=1$	$w=1$	$w=2$	$w=1$	$w=2$	$w=3$
16^2	-/1	-/1	-/1	-/-	-/1	-/-	-/-
32^2	22/22	16/16	36/36	35/1	75/75	-/1	-/1
64^2	22/52	16/16	36/36	36/36	75/119	74/74	73/73
128^2	22/108	16/16	36/36	36/36	75/296	74/74	73/73
256^2	22/217	16/16	36/37	36/36	75/670	74/74	73/73
512^2	22/430	16/16	36/37	36/36	75/1329	74/74	73/73

4.2. Case $x^0 = \pi$

DCT III matrices where the generating function has a root at π appear in solving integral equations, for example the image restoration problems with Neumann (reflecting) boundary conditions, see [6]. According to (6)–(8), if $x^0 = \pi$, then the generating function of the coarser matrix A_k has a unique zero at 0. Therefore, starting from the second coarser level, the problem becomes the one as in §4.1. Using tensor arguments, we can extend the results straightforwardly to the block case where we have $(x_1^0, x_2^0) = (\pi, \pi)$. We test our TGM with $f(x_1, x_2) = 4 + 2\cos x_1 + 2\cos x_2$. Table III gives the results in the block case and confirms the convergence of our two-grid and multigrid method. We have performed the experiments with different kind of data vectors in order to check the independence of the convergence with respect to the possible subspaces where the data vector and the solution belong: we observe that this parameter does not affect the convergence rate.

5. OPEN PROBLEMS AND FUTURE WORK

The numerical evidences reported in the previous section have shown that our MGM (V-cycle) is optimally convergent even when the systems are ill-conditioned. This represents two directions of work that will be of interest both in theory and applications:

- to extend the convergence analysis from TGM to MGM under the assumptions considered in this paper (also making use of the new tools introduced in [1]);
- since the DCT III problems with zeros at (π, π) (or close to (π, π)) are encountered in a variety of image restoration applications (see e.g. [3, 6]), it would be interesting

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Table III. Twogrid/Multigrid - $2D$ Case: A) $(\mathbf{x}_e)_i = i/(n_1 n_2)$, B) $(\mathbf{x}_e)_i = \lfloor i/n_1 \rfloor /n_2 + i \pmod{n_1}/n_1$, C) $(\mathbf{x}_e)_i = \lfloor i/n_1 \rfloor /n_2 + i \pmod{n_1}/n_1 + 10^{-2}(-1)^i$, D) $(\mathbf{x}_e)_i = \lfloor i/n_1 \rfloor /n_2 + i \pmod{n_1}/n_1 + 10^{-1}(-1)^i$, E) $(\mathbf{x}_e)_i = \lfloor i/n_1 \rfloor /n_2 + i \pmod{n_1}/n_1 + (-1)^i$,

$n_1 \cdot n_2$	A	B	C	D	E
16^2	-/1	-/1	-/1	-/1	-/1
32^2	5/5	7/7	7/7	7/7	7/7
64^2	5/5	7/7	7/7	7/7	7/7
128^2	5/4	7/6	7/6	7/6	7/6
256^2	5/4	7/6	7/6	7/6	7/6
512^2	5/4	7/6	7/6	7/6	7/6

to check the robustness of our procedure with respect to noise and the changes in the regularization parameter (for the latter see [4]).

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