Jackson's Theorem and Circulant Preconditioned Toeplitz Systems

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Abstract

Preconditioned conjugate gradient method is used to solve n-by-n Hermitian Toeplitz systems $A_n x = b$. The preconditioner S_n is the Strang's circulant preconditioner which is defined to be the circulant matrix that copies the central diagonals of A_n. The convergence rate of the method depends on the spectrum of $S_n^{-1}A_n$. Using Jackson's theorem in approximation theory, we prove that if A_n has a positive generating function f whose ℓth derivative $f^{(\ell)}$, $\ell \geq 0$, is Lipschitz of order $0 < \alpha \leq 1$, then the method converges superlinearly. We show moreover that the error after $2q$ conjugate gradient steps decreases like $\prod_{k=2}^{q}(\log^2 k/k^{2(\ell+\alpha)})$.

Abbreviated Title. Jackson's Theorem and Toeplitz Systems.

Key Words. Jackson's theorem, Toeplitz matrix, circulant matrix, preconditioned conjugate gradient method, generating function.

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1 Introduction.

An n-by-n matrix $A_n = [a_{i,j}]$ is said to be Toeplitz if $a_{i,j} = a_{i-j}$, i.e. A_n is constant along its diagonals. Toeplitz systems of the form $A_n x = b$ occur in a variety of applications, especially in signal processing and control theory. Existing direct methods for dealing with them include the Levison-Trench-Zohar $O(n^2)$ algorithms [19], and a variety of $O(n \log^2 n)$ algorithms such as the one by Ammar and Gragg [1]. The stability properties of these direct methods for symmetric positive definite matrices are discussed in Bunch [2]. In this paper, we consider an iterative method, the preconditioned conjugate gradient method, for solving Toeplitz systems.

An n-by-n Toeplitz matrix B_n is said to be circulant if its diagonals b_j satisfy $b_{n-j} = b_{-j}$ for $0 < j \le n-1$. We remark that circulant matrices can always be diagonalized by unitary matrices. In fact, we have $B_n = F_n^* \Lambda_n F_n$, where Λ_n is diagonal and F_n is the Fourier matrix with entries given by $[F_n]_{jk} = \frac{1}{\sqrt{n}} e^{\frac{-2\pi ijk}{n}}$, see Davis [11]. Strang [17] first suggested the use of preconditioned conjugate gradient method with circulant matrix B_n as preconditioner for solving positive definite Toeplitz system. Instead of solving $A_n x = b$, we solve the preconditioned system $B_n^{-1} A_n x = B_n^{-1} b$ by the conjugate gradient method with B_n being a circulant matrix.

The number of operations per iteration in the preconditioned conjugate gradient method depends mainly on the work of computing the matrix-vector multiplication $B_n^{-1} A_n y$, see for instance Golub and van Loan [13]. For any vector y, since $B_n^{-1} y = F_n^* \Lambda_n^{-1} F_n y$, the product $B_n^{-1} y$ can be found efficiently by the Fast Fourier Transform in $O(n \log n)$ operations. Likewise, the product $A_n y$ can also be computed by the Fast Fourier Transform by first embedding A_n into a $2n$-by-$2n$ circulant matrix. The multiplication thus requires $O(2n \log(2n))$ operations. It follows that the total operations per iteration is of order $O(n \log n)$. In order to compete with direct methods, the circulant matrix B_n should be chosen such that the conjugate gradient method converges sufficiently fast when applied to the preconditioned system $B_n^{-1} A_n x = B_n^{-1} b$. It is well-known that the method converges fast if $B_n^{-1} A_n$ has a clustered spectrum, i.e. $B_n^{-1} A_n$ is of the form $I_n + U_n + W_n$ where I_n is the identity matrix, U_n is a matrix of low rank and W_n is a matrix of small ℓ_2 norm.

Strang in [17] proposed a possible choice of circulant preconditioner S_n. It is obtained by copying the central diagonals of A_n and reflecting them around to complete the circulant. Chan and Strang [3] then proved that if the diagonals a_j of the Toeplitz matrix A_n are Fourier coefficients of a positive function in the Wiener class, i.e. $\sum_j |a_j| < \infty$, then the eigenvalues of the preconditioned system $S_n^{-1}A_n$ will be clustered around one for large n. It follows that the preconditioned conjugate gradient method, when applied to the preconditioned system, converges superlinearly for large n. More precisely, for all $\epsilon > 0$, there exists a constant $c(\epsilon) > 0$ such that the error vector e_q of the preconditioned conjugate gradient method at the qth iteration satisfies

$$||e_q|| \leq c(\epsilon)\epsilon^q ||e_0|| \tag{1}$$

when n is sufficiently large. Here $||x||^2 = x^* S_n^{-1/2} A_n S_n^{-1/2} x$. Hence the number of iterations required for convergence is independent of the size of the matrix A_n when n is large. In particular, the system $A_n x = b$ can be solved in $O(n \log n)$ operations.

Over the past few years, several other preconditioners have also been proposed, see for instance, T. Chan [9], Chan [5], Tyrtyshinkov [20], Ku and Kuo [16] and Huckle [15]. In Chan [4, 5] and Chan, Jin and Yeung [6], we have shown respectively that the preconditioners proposed in [9], [5] and [20] also work for the Wiener class functions, i.e. (1) holds if $\sum_j |a_j| < \infty$. Huckle, on the other hand, has proved in [15] that his preconditioner works for the class of functions with $\sum_j j|a_j|^2 < \infty$. We remark that it is the Besov space $B_2^{1/2}$. For T. Chan's preconditioner, Chan and Yeung [7] recently have extended the superlinear convergence results to the class of 2π-periodic continuous functions. One of the aims of this paper is to obtain similar results for Strang's preconditioner. We will prove that Strang's preconditioner works for a slightly smaller class of functions (see (30)) than T. Chan's preconditioner does.

In the conjugate gradient method, an estimate of the number of iterations required for convergence can be obtained by studying the precise rate at which $||e_q||$ goes to zero in (1). Trefethen [18] first proved that if f is a positive rational function of type (μ, ν), then the preconditioned system $S_n^{-1}A_n$ has at most $1 + 2\max\{\mu, \nu\}$ distinct eigenvalues. Hence the conjugate gradient method, when applied to the preconditioned system, converges in at most $1 + 2\max\{\mu, \nu\}$ steps. He also proved that if f is positive and analytic in a neighbourhood of $|z| = 1$ and if S_n is used as preconditioner, then there exist constants $c > 0$ and $0 \le r < 1$ such that

$$||e_q|| \le c r^{q^2} ||e_0||$$

for n sufficiently large. His proof uses the theory of Carathéodory-Fejér approximation to approximate the singular values of a Hankel matrix which is obtained from $S_n^{-1} A_n$ by an orthogonal transformation.

In Chan [5], we considered functions f that are less smooth, and using tools in linear algebra, we proved the following result.

Theorem 1 *Let f be a positive ν-times differentiable function with its νth derivative in $L^1[0, 2\pi]$, where $\nu > 1$. If S_n is used as preconditioner for A_n, then for large n,*

$$||e_{2q}|| \le \frac{c^q}{((q-1)!)^{2\nu-2}} ||e_0||, \tag{2}$$

for some constant c that depends on f and ν only.

Another aim of this paper is to improve the above result and to extend it to the class of Lipschitz functions of order $\nu > 0$. Our main tool is Jackson's theorem in polynomial approximation. We will show that for a positive function f whose ℓth derivative $f^{(\ell)}$, $\ell \ge 0$, is Lipschitz with order $0 < \alpha \le 1$, the error vector e_{2q} is bounded by

$$||e_{2q}|| \le \prod_{k=2}^{q} \left(\frac{c \log k}{k^{(\ell+\alpha)}} \right)^2 ||e_0||, \tag{3}$$

where c is a constant that depends only on f. For $\nu = \ell + \alpha > 1$, (3) can be rewritten as

$$||e_{2q}|| \le \frac{c^q}{((q-1)!)^{2\nu-2}} ||e_0|| \cdot \left\{ \frac{1}{q^{2\nu-2}} \prod_{k=2}^{q} \frac{\log^2 k}{k^2} \right\}.$$

Since

$$\frac{1}{q^{2\nu-2}} \prod_{k=2}^{q} \frac{\log^2 k}{k^2} < 1$$

for all q, we see that (3) is a better bound than (2).

This paper is organized as follows. In §2, some results in approximation theory are introduced and the spectra of A_n and S_n are analysed. In §3, we find the distribution of the eigenvalues of $A_n - S_n$ and show that they are clustered around zero. In §4, we analyse the spectrum of $S_n^{-1}A_n$ and use the results to derive the bound (3) for $||e_{2q}||$. Some concluding remarks are given in §5.

2 The Spectra of A_n and S_n.

To begin with, let $\mathcal{C}_{2\pi}$ be the Banach space of all 2π-periodic continuous real-valued functions defined on the real line $\mathbf{R}$ and equipped with the supremum norm $||\cdot||_\infty$. Let

$$a_k(f) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-ik\theta}d\theta, \quad k = 0, \pm 1, \pm 2, \cdots,$$

be the Fourier coefficients of f for f in $\mathcal{C}_{2\pi}$. We remark that for all integers k, $a_{-k} = \bar{a}_k$ as f is real-valued. Let $A_n(f)$ be the n-by-n Hermitian Toeplitz matrix with the (j,l)th entry given by $a_{j-l}(f)$. The function f is called the generating function of the matrices $A_n(f)$. The following Lemma, proved in Grenander and Szegö [14], gives the relation between f and the spectrum $\sigma(A_n(f))$ of $A_n(f)$. For simplicity, we let $f_{\min}$ and $f_{\max}$ be the minimum and maximum values of f. Thus

$$f_{\min} \le f(\theta) \le f_{\max}, \quad \forall \theta \in \mathbf{R}.$$

Lemma 1 *Let $f \in \mathcal{C}_{2\pi}$. Then the spectrum $\sigma(A_n(f))$ of $A_n(f)$ satisfies*

$$\sigma(A_n(f)) \subseteq [f_{\min}, f_{\max}], \quad \forall n \ge 1. \tag{4}$$

In particular, we have

$$||A_n(f)||_2 \le ||f||_\infty, \quad \forall n \ge 1. \tag{5}$$

If, moreover, f is a positive function, i.e., $f_{\min} > 0$, then $A_n(f)$ is positive definite for all n.

Given $A_n(f)$, Strang's preconditioner $S_n(f)$ is defined to be the circulant matrix that copies the central diagonals of $A_n(f)$ and reflects them around to complete the circulant. More precisely, the kth entry in the first column of $S_n(f)$ is given by

$$[S_n(f)]_{k,0} = \begin{cases} a_k(f) & 0 \le k \le m, \\ a_{k-n}(f) & m < k < n. \end{cases} \tag{6}$$

Here and in the following, we assume for simplicity that $n = 2m + 1$. If $n = 2m$, we define $[S_n(f)]_{m,0} = 0$.

We prove below that the eigenvalues of $S_n(f)$ are given by the mth partial sum $s_m(f)$ of f at equally-spaced points in $[0, 2\pi]$. We recall that the jth partial sum of f is defined as

$$s_j(f)(\theta) \equiv \sum_{k=-j}^{j} a_k(f) e^{ik\theta}, \qquad \forall \theta \in \mathbf{R}.$$

Lemma 2 *The eigenvalues $\lambda_j(S_n(f))$ of $S_n(f)$ are given by*

$$\lambda_j(S_n(f)) = \sum_{k=-m}^{m} a_k(f) e^{2\pi ijk/n} = s_m(f)(\frac{2\pi j}{n}), \qquad 0 \le j \le n-1. \tag{7}$$

In particular,

$$||S_n(f)||_2 \le ||s_m(f)||_\infty, \quad \forall n \ge 1. \tag{8}$$

Proof: Since $S_n(f)$ is circulant, its eigenvalues are given in terms of the entries in the first column:

$$\lambda_j(S_n(f)) = \sum_{k=0}^{n-1} [S_n(f)]_{k,0} e^{2\pi ijk/n}, \quad 0 \le j \le n-1,$$

see Davis [11] for instance. Using (6), this becomes

$$\begin{aligned} \lambda_j(S_n(f)) &= \sum_{k=0}^{m} a_k(f) e^{2\pi ijk/n} + \sum_{k=m+1}^{n-1} a_{k-n}(f) e^{2\pi ijk/n} \\ &= \sum_{k=-m}^{m} a_k(f) e^{2\pi ijk/n} = s_m(f)(\frac{2\pi j}{n}), \quad 0 \le j \le n-1. \quad \Box \end{aligned}$$

Our next aim is to give similar bounds on the spectrum of $S_n(f)$ as in (4) and (5). We first recall some definitions and theorems in approximation theory. For all $k > 0$, let

$$\mathcal{P}_k \equiv \{f \in \mathcal{C}_{2\pi} \mid f(\theta) = \sum_{j=-k}^{k} \rho_j e^{ij\theta}, \ \ \rho_j = \bar{\rho}_{-j}, \ \ 0 \le j \le k\}$$

be the space of all kth degree real-valued trigonometric polynomials. The best polynomial approximation of f is measured by

$$E_k(f) = \inf_{p_k \in \mathcal{P}_k} ||f - p_k||_\infty.$$

Since $\mathcal{P}_k$ is finite dimensional, it is clear that the infimum can always be attained by some polynomials in $\mathcal{P}_k$.

For $0 < \alpha \le 1$, let

$$\mathcal{L}_\alpha \equiv \{f \in \mathcal{C}_{2\pi} \mid \sup_{\theta_1 \neq \theta_2} \frac{|f(\theta_1) - f(\theta_2)|}{|\theta_1 - \theta_2|^\alpha} < \infty\}$$

be the space of all Lipschitz functions of order α. For all $\nu > 0$, we write $\nu = \ell + \alpha$ where ℓ is a non-negative integer and $0 < \alpha \le 1$. Then we define

$$\mathcal{C}_{2\pi}^{\nu} = \{f \in \mathcal{C}_{2\pi} \mid f^{(\ell)} \in \mathcal{L}_\alpha\}.$$

The following two Lemmas relate $||s_k(f) - f||_\infty$ with $E_k(f)$ for functions in $\mathcal{C}_{2\pi}^{\nu}$. Their proofs can be found in Cheney [10] and Feinerman and Newman [12].

Lemma 3 *(Dini-Lipschitz Theorem) For all $f \in \mathcal{C}_{2\pi}$, we have*

$$||s_k(f) - f||_\infty \le (4 + \log k) E_k(f), \quad \forall k \ge 1. \tag{9}$$

Lemma 4 *(Jackson's Theorem) For all $f \in \mathcal{C}_{2\pi}^{\nu}$, $\nu > 0$, we have*

$$E_k(f) \le \frac{c}{(k+1)^\nu}, \quad \forall k \ge 1, \tag{10}$$

where c is a constant that depends only on f and ν.

As a corollary to the two Lemmas above, we give a bound on the spectrum of $S_n(f)$ for positive functions $f \in \mathcal{C}_{2\pi}^{\nu}$, $\nu > 0$.

Theorem 2 *Let f be a positive function in $\mathcal{C}_{2\pi}^{\nu}$ with $\nu > 0$. Then for large n, the spectrum $\sigma(S_n(f))$ of $S_n(f)$ satisfies*

$$\sigma(S_n(f)) \subseteq [\frac{1}{2} f_{\min}, 2 f_{\max}]. \tag{11}$$

In particular, $S_n(f)$ is positive definite and

$$||S_n^{-1}(f)||_2 \leq \frac{2}{f_{\min}}. \tag{12}$$

Proof: By (7), the eigenvalues of $S_n(f)$ are given by

$$\lambda_j(S_n(f)) = s_m(f)(\frac{2\pi j}{n}) = (s_m(f) - f)(\frac{2\pi j}{n}) + f(\frac{2\pi j}{n}), \quad 0 \leq j \leq n-1.$$

Thus if $\lambda_{\max}(S_n(f))$ is the largest eigenvalue of $S_n(f)$, then by (9) and (10), we have

$$\begin{aligned} \lambda_{\max}(S_n(f)) &\leq ||s_m(f) - f||_\infty + f_{\max} \\ &\leq (4 + \log m) E_m(f) + f_{\max} \\ &\leq c \frac{4 + \log m}{(m+1)^\nu} + f_{\max}, \end{aligned}$$

where c depends only on f. Since $f_{\max} \geq f_{\min} > 0$ and

$$\lim_{m \to \infty} \frac{4 + \log m}{(m+1)^\nu} = 0$$

for $\nu > 0$, it follows that for $n = 2m+1$ sufficiently large,

$$\lambda_{\max}(S_n(f)) \leq 2 f_{\max}.$$

Similarly, the smallest eigenvalue $\lambda_{\min}(S_n(f))$ of $S_n(f)$ is bounded below by

$$\begin{aligned} \lambda_{\min}(S_n(f)) &\geq f_{\min} - ||s_m(f) - f||_\infty \\ &\geq f_{\min} - (4 + \log m) E_m(f) \\ &\geq f_{\min} - c \frac{4 + \log m}{(m+1)^\nu}. \end{aligned}$$

Since $f_{\min} > 0$, we have, for n sufficiently large,

$$\lambda_{\min}(S_n(f)) \geq \frac{1}{2} f_{\min}. \quad \square$$

3 The Spectrum of $A_n(f) - S_n(f)$.

In this section we show that for Strang's preconditioner defined by (6), the spectrum of $A_n(f) - S_n(f)$ will be clustered around zero provided that $f \in \mathcal{C}_{2\pi}^{\nu}$, $\nu > 0$. We begin with the following Lemma.

Lemma 5 *Let $p_k \in \mathcal{P}_k$ for some $k \leq m$. Then $A_n(p_k) - S_n(p_k)$ can be written as*

$$A_n(p_k) - S_n(p_k) = U(p_k) - V(p_k), \tag{13}$$

where $U(p_k)$ and $V(p_k)$ are positive semi-definite matrices of rank at most k.

Proof: Since p_k is a degree k real trigonometric polynomial and $S_n(p_k)$ copies the central diagonals of $A_n(p_k)$, it is clear that the $(2m+1)$-by-$(2m+1)$ matrix $A_n(p_k) - S_n(p_k)$ is Hermitian and of the form

$$A_n(p_k) - S_n(p_k) = \begin{bmatrix} 0 & 0 & B \\ 0 & 0 & 0 \\ B^* & 0 & 0 \end{bmatrix}, \tag{14}$$

where B is an m-by-m Toeplitz matrix with at most k nonzero diagonals at the upper right hand corner. Let $P\Lambda Q^*$ be the singular value decomposition of B, see Golub and van Loan [13]. Thus P and Q are m-by-m unitary matrices and Λ is a nonnegative diagonal matrix with at most k positive diagonal entries. It is straightforward to check that

$$\frac{1}{\sqrt{2}} \begin{bmatrix} P & 0 & P \\ 0 & \sqrt{2} & 0 \\ Q & 0 & -Q \end{bmatrix},$$

is an n-by-n unitary matrix and

$$\frac{1}{2} \begin{bmatrix} P^* & 0 & Q^* \\ 0 & \sqrt{2} & 0 \\ P^* & 0 & -Q^* \end{bmatrix} \begin{bmatrix} 0 & 0 & B \\ 0 & 0 & 0 \\ B^* & 0 & 0 \end{bmatrix} \begin{bmatrix} P & 0 & P \\ 0 & \sqrt{2} & 0 \\ Q & 0 & -Q \end{bmatrix} = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\Lambda \end{bmatrix}.$$

Hence except for a single zero eigenvalue, all eigenvalues of $A_n(p_k) - S_n(p_k)$ occur in pairs $\pm\lambda$, where each λ is a singular value of B. $\Box$

Thus if $f \in \mathcal{P}_k$, then $A_n(f) - S_n(f)$ has at most $2k$ nonzero eigenvalues. For functions $f \in \mathcal{C}_{2\pi}$, we have the following Lemma.

Lemma 6 *Let $f \in \mathcal{C}_{2\pi}$. Then for all $n \geq 1$, we have*

$$A_n(f) - S_n(f) = U_k(f) - V_k(f) + W_k(f), \quad 1 \leq k \leq m,$$

where $U_k(f)$ and $V_k(f)$ are positive semi-definite matrices of rank at most k and

$$||W_k(f)||_2 \leq (4 + \log m)E_m(f) + 2E_k(f), \quad 1 \leq k \leq m. \tag{15}$$

Proof: For all $1 \leq k \leq m$, let $p_k \in \mathcal{P}_k$ be the best approximation of f in $\mathcal{P}_k$, i.e.

$$||f - p_k||_\infty = E_k(f). \tag{16}$$

Clearly

$$A_n(f) - S_n(f) = A_n(p_k) - S_n(p_k) + A_n(f - p_k) - S_n(f - p_k). \tag{17}$$

By (13), the first two terms in the right hand side of (17) can be written as

$$A_n(p_k) - S_n(p_k) = U_k - V_k,$$

where U_k and V_k are positive semi-definite matrices of rank at most k. It remains to find a bound in ℓ_2 norm for the last two terms in (17). By (5), we have

$$||A_n(f - p_k)||_2 \leq ||f - p_k||_\infty = E_k(f). \tag{18}$$

By (8), the triangle inequality, (9) and (16), we have

$$\begin{aligned} ||S_n(f - p_k)||_2 &\leq ||s_m(f - p_k)||_\infty = ||s_m(f) - p_k||_\infty \\ &\leq ||s_m(f) - f||_\infty + ||f - p_k||_\infty \\ &\leq (4 + \log m)E_m(f) + E_k(f). \end{aligned} \tag{19}$$

Putting (18) and (19) into (17), we get (15). $\Box$

Using Cauchy's interlace theorem, see Wilkinson [21], we see that except for the $2k$ outlying eigenvalues, all the eigenvalues of $A_n(f) - S_n(f)$ are in the neighborhood of zero with radius equal to $(4 + \log m)E_m(f) + 2E_k(f)$. Our next task is to estimate the radius.

Lemma 7 *Let $\nu > 0$. Then for m sufficiently large, we have*

$$\frac{\log m}{m^\nu} \leq \frac{\log k}{k^\nu}, \quad 2 \leq k \leq m. \tag{20}$$

Proof: Consider the function

$$g(x) = \frac{\log x}{x^\nu}, \quad x \geq 2.$$

Its derivative is given by

$$g'(x) = \frac{1 - \nu \log x}{x^{\nu+1}}.$$

Thus $g'(x) \leq 0$ if and only if $x \geq e^{1/\nu}$. Hence when $2 \geq e^{1/\nu}$, $g(x)$ is a decreasing function for all $x \geq 2$. In particular, (20) holds for all m.

If $e^{1/\nu} \geq 2$, then $g(x)$ is an increasing function for $2 \leq x \leq e^{1/\nu}$ and a decreasing function for $x \geq e^{1/\nu}$, i.e.

$$\frac{\log 2}{2^\nu} \leq \frac{\log x}{x^\nu}, \qquad 2 \leq x \leq e^{1/\nu},$$

and

$$\frac{\log m}{m^\nu} \leq \frac{\log x}{x^\nu}, \qquad e^{1/\nu} \leq x \leq m,$$

Since $g(x) \to 0$ as $x \to \infty$, we see that for all m sufficiently large,

$$\frac{\log m}{m^\nu} \leq \frac{\log 2}{2^\nu}.$$

Hence (20) follows. $\square$

As an immediate corollary to the above Lemma, we have the following result.

Lemma 8 *Let $f \in \mathcal{C}_{2\pi}^\nu$ with $\nu > 0$. Then there exists a constant c which depends only on f and ν such that*

$$(4 + \log m)E_m(f) + 2E_k(f) \leq c\frac{\log(k+1)}{(k+1)^\nu}, \qquad 1 \leq k \leq m,$$

for all m sufficiently large.

Proof: Since $2 \leq 3\log(k+1)$ for all $k \geq 1$, we have, for large m,

$$(4 + \log m)E_m(f) + 2E_k(f) \leq 2\log(m+1)E_m(f) + 3\log(k+1)E_k(f),$$

for $1 \leq k \leq m$. Thus for f in $\mathcal{C}_{2\pi}^{\nu}$, by (10) and (20), we have for large m,

$$\begin{aligned} & (4+\log m)E_m(f)+2E_k(f) \\ \leq\ & 2\log(m+1)\frac{c}{(m+1)^{\nu}}+3\log(k+1)\frac{c}{(k+1)^{\nu}}, \\ \leq\ & 5c\frac{\log(k+1)}{(k+1)^{\nu}}, \quad 1 \leq k \leq m. \quad \square \end{aligned}$$

Combining Lemmas 6 and 8, we have our main theorem of this section.

Theorem 3 *Let $f \in \mathcal{C}_{2\pi}^{\nu}$ with $\nu > 0$. Then for n large,*

$$A_n(f) - S_n(f) = U_k(f) - V_k(f) + W_k(f), \qquad 1 \leq k \leq m, \tag{21}$$

where $U_k(f)$ and $V_k(f)$ are positive semi-definite matrices of rank at most k and

$$||W_k(f)||_2 \leq \frac{c\log(k+1)}{(k+1)^{\nu}}, \qquad 1 \leq k \leq m. \tag{22}$$

Here c is a constant that depends only on f and ν.

As an immediate corollary, we can prove that the spectrum of $A_n(f) - S_n(f)$ is clustered around zero.

Theorem 4 *Let $f \in \mathcal{C}_{2\pi}^{\nu}$ with $\nu > 0$. Then for all $\epsilon > 0$, there exists a $K > 0$ such that for all $n \geq 1$, at most $2K$ eigenvalues of $A_n(f) - S_n(f)$ have absolute values exceeding ϵ.*

Proof: Let $M > 0$ be chosen such that (21) and (22) hold for all $m > M$. For all $\epsilon > 0$, let $K \geq M$ be chosen such that

$$\frac{c\log(K+1)}{(K+1)^{\nu}} \leq \epsilon,$$

where c is given in (22). Then for $1 \leq m \leq K$, since the $(m+1)$st row of $A_n(f) - S_n(f)$ is zero (cf. (14)), $A_n(f) - S_n(f)$ has at most $n-1 = 2m \leq 2K$ nonzero eigenvalues. For $m \geq K$, we apply Cauchy's interlace theorem to (21) with $k = K$, then we see that at most $2K$ eigenvalues of $A_n(f) - S_n(f)$ have absolute values exceeding ϵ. $\square$

4 The Spectrum of $S_n^{-1}(f)A_n(f)$.

We begin by showing that the spectrum of $S_n^{-1}(f)A_n(f)$ is clustered around one.

Theorem 5 *Let $f \in \mathcal{C}_{2\pi}^{\nu}$ with $\nu > 0$. If f is positive, then for all $\epsilon > 0$, there exists a $K > 0$, such that for all n sufficiently large, at most $2K$ eigenvalues of $S_n^{-1}(f)A_n(f) - I_n$ have absolute values larger than ϵ.*

Proof: Since f is positive, by Theorem 2, there exists an $N > 0$ such that for all $n > N$, $S_n(f)$ is positive definite and $S_n^{-1}(f)$ is bounded uniformly in the ℓ_2 norm. Since

$$S_n^{-1}(f)A_n(f) - I_n = S_n^{-1}(f)(A_n(f) - S_n(f)),$$

and by Theorem 4, $A_n(f) - S_n(f)$ has clustered spectrum around zero, it follows that $S_n^{-1}(f)A_n(f) - I_n$ has clustered spectrum around zero for all $n > N$. $\Box$

Using Theorem 5, one can conclude easily, see Chan and Strang [3] for instance, that the conjugate gradient method converges superlinearly when applied to the preconditioned system $S_n^{-1}(f)A_n(f)$, i.e. (1) holds for all $\epsilon > 0$ provided that n is sufficiently large. In the following, we derive an estimate of the rate at which the norm $||e_q||$ of the error vector at the qth iteration converges to zero.

Theorem 6 *Let $f \in \mathcal{C}_{2\pi}^{\nu}$ with $\nu > 0$. If f is positive, then for large n,*

$$||e_{2q}|| \leq \prod_{k=2}^{q} \frac{c \log^2 k}{k^{2\nu}} ||e_0||, \qquad 1 \leq q \leq m, \tag{23}$$

for some constant c that depends on f and ν only.

Proof: We remark that from the standard error analysis of the conjugate gradient method, we have

$$||e_q|| \leq [\, \min_{P_q} \max_{\lambda} |P_q(\lambda)| \,] \; ||e_0||, \tag{24}$$

where the minimum is taken over polynomials of degree q with constant term 1 and the maximum is taken over the spectrum of $S_n^{-1}(f)A_n(f)$, or equivalently, the spectrum of $S_n^{-1/2}(f)A_n(f)S_n^{-1/2}(f)$, see for instance, Golub and van Loan [13]. In the following, we will try to estimate that minimum. For simplicity, we write $A_n(f)$ and $S_n(f)$ as A_n and S_n respectively.

Let $\hat{B}_n \equiv S_n^{-1/2}(A_n - S_n)S_n^{-1/2}$. Then by (21), we have for large n,

$$\begin{aligned} \hat{B}_n &= S_n^{-1/2}U_kS_n^{-1/2} - S_n^{-1/2}V_kS_n^{-1/2} + S_n^{-1/2}W_kS_n^{-1/2} \\ &\equiv \hat{U}_k - \hat{V}_k + \hat{W}_k, \quad 1 \le k \le m. \end{aligned} \tag{25}$$

Clearly $\hat{U}_k$ and $\hat{V}_k$ are still positive semi-definite matrices of rank at most k. By (12) and (22)

$$||\hat{W}_k||_2 \le ||S_n^{-1}||_2||W_k||_2 \le \frac{\hat{c}\log(k+1)}{(k+1)^\nu}, \quad 1 \le k \le m, \tag{26}$$

with $\hat{c} = 2c/f_{\min}$.

Let us order the eigenvalues of $\hat{B}_n$ as

$$\mu_0^- \le \mu_1^- \le \cdots \le 0 \le \cdots \le \mu_1^+ \le \mu_0^+.$$

By applying Cauchy's interlace theorem to (25) and using the bound of $||\hat{W}_k||_2$ in (26), we see that for $1 \le k \le m$, there are at most k eigenvalues of $\hat{B}_n$ lying to the right of $\hat{c}\log(k+1)/(k+1)^\nu$ and there are at most k of them lying to the left of $-\hat{c}\log(k+1)/(k+1)^\nu$. More precisely, we have

$$|\mu_k^\pm| \le ||\hat{W}_k||_2 \le \frac{\hat{c}\log(k+1)}{(k+1)^\nu}, \quad 1 \le k \le m.$$

Using the identity

$$S_n^{-1/2}A_nS_n^{-1/2} = I_n + \hat{B}_n,$$

we see that if we order the eigenvalues of $S_n^{-1/2}A_nS_n^{-1/2}$ as

$$\lambda_0^- \le \lambda_1^- \le \cdots \le 0 \le \cdots \le \lambda_1^+ \le \lambda_0^+,$$

then $\lambda_k^\pm = 1 + \mu_k^\pm$ for all $k \ge 0$ with

$$1 - \frac{\hat{c}\log(k+1)}{(k+1)^\nu} \le \lambda_k^- \le \lambda_k^+ \le 1 + \frac{\hat{c}\log(k+1)}{(k+1)^\nu}, \quad 1 \le k \le m. \tag{27}$$

For $\lambda_0^{\pm}$, the bounds are obtained from (4) and (11). In fact, we have

$$\frac{f_{\min}}{2f_{\max}} \le \lambda_k^- \le \lambda_k^+ \le \frac{2f_{\max}}{f_{\min}}, \quad \forall k \ge 0. \tag{28}$$

Having obtained the bounds for $\lambda_k^{\pm}$, we can now construct the polynomial that will give us a bound for (24). Our idea is to choose P_{2q} that annihilates the q extreme pairs of eigenvalues. Thus consider

$$p_k(x) = (1 - \frac{x}{\lambda_k^+})(1 - \frac{x}{\lambda_k^-}), \quad 1 \le k \le m.$$

Between those roots $\lambda_k^{\pm}$, the maximum of $|p_k(x)|$ is attained at the average $x = \frac{1}{2}(\lambda_k^+ + \lambda_k^-)$, where by (27) and (28), we have

$$\begin{aligned}
\max_{x \in [\lambda_k^-, \lambda_k^+]} |p_k(x)| &= \frac{(\lambda_k^+ - \lambda_k^-)^2}{4\lambda_k^+ \lambda_k^-} \\
&\le \left(\frac{2\hat{c}\log(k+1)}{(k+1)^{\nu}}\right)^2 \cdot (\frac{f_{\max}}{f_{\min}})^2 \\
&= \left(\frac{2\hat{c} f_{\max}}{f_{\min}}\right)^2 \cdot \frac{\log^2(k+1)}{(k+1)^{2\nu}}, \quad 1 \le k \le m,
\end{aligned}$$

Similarly, for $k = 0$, we have, by using (28),

$$\max_{x \in [\lambda_0^-, \lambda_0^+]} |p_0(x)| = \frac{(\lambda_0^+ - \lambda_0^-)^2}{4\lambda_0^+ \lambda_0^-} \le \frac{(4f_{\max}^2 - f_{\min}^2)^2}{4f_{\min}^4}.$$

Hence the polynomial $P_{2q} = p_0 p_1 \cdots p_{q-1}$, which annihilates the q extreme pairs of eigenvalues, satisfies

$$\max_{x \in [\lambda_{q-1}^-, \lambda_{q-1}^+]} |P_{2q}(x)| \le \prod_{k=2}^{q} \frac{c \log^2 k}{k^{2\nu}}, \quad 1 \le q \le m. \tag{29}$$

Here c is some constant that depends only on f and ν. Since the remaining $n - 2q$ eigenvalues $\{\lambda_k^{\pm}\}_{k \ge q}$ are in the interval $[\lambda_{q-1}^-, \lambda_{q-1}^+]$, (23) now follows directly from (24) and (29). $\Box$

5 Concluding Remarks.

We first remark that for Strang's preconditioner, we can use the technique presented here to prove the superlinear convergence of the method for a larger class of functions, namely the class of functions f that satisfy

$$\lim_{m\to\infty} \log m \cdot E_m(f) = 0, \tag{30}$$

cf (15). However, for this class of functions, we can only obtain the bound (1) for $||e_q||$. This is to be compared with T. Chan's preconditioner where (1) holds even for $f \in \mathcal{C}_{2\pi}$. We also remark that our results cannot be generalized readily to the class of positive functions in $\mathcal{C}_{2\pi}$. This is because for $f \in \mathcal{C}_{2\pi}$, $s_m(f)$ may not converge to f uniformly in $\mathbf{R}$. Hence we cannot conclude as in (11) that $\lambda_{\min}(S_n(f)) \geq \frac{1}{2} f_{\min}$. Finally we remark that although the results we proved here are asymptotic results that hold when n is large, in practice, the method converges superlinearly for n that are small as well, see the numerical results in Chan [5] for instance.

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