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Generalization of Strang's Preconditioner with Applications to Toeplitz Least Squares Problems

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In this paper, we propose a method to generalize Strang's circulant preconditioner for arbitrary n-by-n matrices A_n. The $\lfloor \frac{n}{2} \rfloor$th column of our circulant preconditioner S_n is equal to the $\lfloor \frac{n}{2} \rfloor$th column of the given matrix A_n. Thus if A_n is a square Toeplitz matrix, then S_n is just the Strang circulant preconditioner. When S_n is not Hermitian, our circulant preconditioner can be defined as $(S_n^* S_n)^{1/2}$. This construction is similar to the forward-backward projection method used in constructing preconditioners for tomographic inversion problems in medical imaging. We show that if the matrix A_n has decaying coefficients away from the main diagonal, then $(S_n^* S_n)^{1/2}$ is a good preconditioner for A_n. Comparisons of our preconditioner with other circulant-based preconditioners are carried out for some 1–D Toeplitz least squares problems: $\min \|b - Ax\|_2$. Preliminary numerical results show that our preconditioner performs quite well, in comparison to other circulant preconditioners. Promising test results are also reported for a 2–D deconvolution problem arising in ground–based atmospheric imaging.

KEY WORDS Toeplitz least squares problems, circulant preconditioned conjugate gradient method, deconvolution, image restoration, atmospheric imaging, medical imaging.

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1. Introduction

An n-by-n matrix A_n is called a Toeplitz matrix if its entries are constant along each diagonal, i.e.

$$A_n = [a_{j,k}]_{0 \le j,k \le n-1} = [a_{j-k}]_{0 \le j,k \le n-1}.$$

In 1986, Strang [22] addressed the question of whether iterative methods can compete with direct methods for solving symmetric positive definite Toeplitz systems of linear equations. The answer has turned out to be an unqualified yes. Strang proposed the use of circulant matrices to precondition Toeplitz matrices in conjugate gradient iterations. Strang's circulant preconditioner S_n is defined to be the matrix that copies the central diagonals of A_n and reflects them around to complete the circulant. More precisely, the entries in the first column of S_n are given by

$$[S_n]_{k,0} = \begin{cases} a_k, & 0 \le k \le \lfloor \frac{n}{2} \rfloor, \\ a_{n-k}, & \lfloor \frac{n}{2} \rfloor < k < n. \end{cases} \tag{1.1}$$

(Here $\lfloor \frac{n}{2} \rfloor$ denotes the largest integer $m \le \frac{n}{2}$.)

The reason why this approach is competitive with direct methods is clear. The use of circulant matrices as preconditioners for Toeplitz problems allows to use the fast Fourier transforms (FFTs) throughout the computations, and these FFT–based iterations are not only numerically efficient, but also highly parallelizable. The convergence rate of the method has been analyzed by Chan and Strang [3]. They proved that if the diagonals of the Toeplitz matrix are Fourier coefficients $\{a_j\}_{j=-\infty}^{\infty}$ of a positive function in Wiener class (i.e. $\{a_j\}_{j=-\infty}^{\infty}$ is absolutely summable), then the spectrum of the circulant preconditioned system $S_n^{-1}A_n$ is clustered around 1 and the preconditioned conjugate gradient method converges superlinearly. More precisely, for any given $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$ such that the error vector e_q at the qth iteration satisfies

$$|||e_q||| \le C(\epsilon)\epsilon^q |||e_0|||, \tag{1.2}$$

where

$$|||x||| \equiv x^* S_n^{-1/2} A_n S_n^{-1/2} x, \tag{1.3}$$

see Chan and Strang [3]. It follows that the preconditioned iterations converge very fast.

Besides Strang's initial circulant preconditioner, several other successful circulant preconditioners have been proposed and analyzed, see, e.g. T. Chan [9], Huckle [16], Ku and Kuo [18], and Tyrtyshnikov [23]. In these papers, it has been shown that under the same assumptions on the Toeplitz matrices, these circulant preconditioned systems also converge superlinearly. Among these preconditioners, we remark that the T. Chan's circulant preconditioner is defined for general square matrices, not necessarily of Toeplitz form. Given any general n-by-n matrix A_n, it is defined to be the minimizer of $\|Q_n - A\|_F$ over all n-by-n circulant matrices Q_n. (Here $\|\cdot\|_F$ denotes the Frobenius norm.) Most circulant preconditioners including Strang's, are not defined for arbitrary matrices. Therefore, T. Chan's preconditioner is particularly useful in solving *non-Toeplitz* systems arising from the numerical solution of elliptic partial differential equations [5] and Toeplitz least squares problems arising from signal and image processing [6,7,10,21]. Convergence results for T. Chan's preconditioner have also been established for these problems.

In this paper, we propose a method to generalize the construction of Strang's circulant preconditioner to arbitrary n-by-n matrices A_n and to employ this new circulant approximation in deconvolution applications in signal and image processing. The idea of constructing such a preconditioner is simple. Given A_n, the $\lfloor \frac{n}{2} \rfloor$th column of our circulant preconditioner S_n is equal to the $\lfloor \frac{n}{2} \rfloor$th column of A_n. Thus if A_n is a square Toeplitz matrix, then S_n is just the Strang circulant preconditioner. When S_n is not Hermitian, our circulant preconditioner can be defined as $(S_n^* S_n)^{1/2}$. For matrices of low displacement rank, there exists another generalization of Strang's preconditioner, see Freund and Huckle [12].

It turns out that the idea of constructing an approximation by selecting the central column of a given matrix has been considered in tomographic inversion problems in image processing and has been referred to as the forward–backward projection method [11]. The approximation matrix is used as a preconditioner to speed up the convergence of the steepest descent method. One of the purposes of this paper is to analyze the convergence properties of this approximating matrix when used as a preconditioner in the preconditioned conjugate gradient method. We show that if the matrix A_n has decaying coefficients away from the main diagonal, then the circulant approximating matrix is a good preconditioner for A_n and hence we expect fast convergence when applying the preconditioned conjugate gradient method to solve these problems. Numerical tests are given to illustrate fast convergence.

The outline of the paper is as follows. In §2, we define our generalized Strang preconditioner and discuss it in relation to the forward-backward projection method [11]. We also mention some of the standard results on circulant preconditioned Toeplitz systems. In §3, we study some applications of our generalized Strang circulant preconditioner to Toeplitz least squares problems and deconvolution problems, and analyze the convergence rate of the preconditioned systems for these applications. In §4, some numerical results are reported, including comparisons with the block-based and displacement-based preconditioning schemes suggested in [6,8]. Test results are also reported for a 2–D Toeplitz least squares deconvolution problem arising from ground-based atmospheric imaging, which is also considered in [19,20] using an inverse Toeplitz preconditioner.

2. Generalized Strang Preconditioner

In the convergence analysis of circulant preconditioned conjugate iterations for Toeplitz systems, one often considers Toeplitz matrices A_n that are generated by a fixed function. More precisely, we assume that there is a function f given by

$$f(\theta) = \sum_{k=-\infty}^{\infty} a_k e^{-ik\theta}, \quad \forall \theta \in [0, 2\pi]$$

such that the n-by-n Toeplitz matrices A_n under consideration have their diagonals given by $\{a_j\}_{j=-n+1}^{n-1}$. The eigenvalues in the spectrum of A_n are closely related to the functional values of its generating function f, as indicated in the following theorem.

Lemma 2.1. (Grenander and Szegö [14]) *Let $f(\theta) \in L_2[0, 2\pi]$ be real-valued and the generating function of the sequence of square Toeplitz matrices A_n. Then the spectra of A_n are uniformly bounded in the interval $[f_{\inf}, f_{\sup}]$ where $f_{\inf}$ and $f_{\sup}$ are the essential infimum and supremum of $f(\theta)$ respectively.*

Given a Toeplitz matrix A_n with diagonals $\{a_j\}_{j=-n+1}^{n-1}$, its Strang circulant preconditioner S_n is defined as in (1.1). We note that the spectrum of S_n is also closely related to f. In fact, we have

$$\lambda_k(S_n) = \sum_{|j| \le \lfloor n/2 \rfloor} a_j e^{2\pi ijk/n}, \quad 0 \le k \le n-1, \tag{2.4}$$

see for instance Chan and Yeung [4]. For a general non-Toeplitz matrix $A_n = [a_{j,k}]$, we define its generalized Strang circulant preconditioner S_n as

$$[S_n]_{k,\lfloor \frac{n}{2} \rfloor} = a_{k,\lfloor \frac{n}{2} \rfloor}, \quad 0 \le k \le n-1,$$

i.e., the $\lfloor \frac{n}{2} \rfloor$th column of S_n is given by the $\lfloor \frac{n}{2} \rfloor$th column of A_n. Note that if A_n is Toeplitz, then this definition coincides with that in (1.1). In general S_n constructed from this scheme will not be Hermitian even if A_n is Hermitian. In this case, we define the generalized Strang preconditioner as $(S_n^* S_n)^{1/2}$.

We remark that our idea of constructing the generalized Strang circulant preconditioner is the same as the forward-backward projection method discussed in [11] for image reconstruction. That method tries to estimate the point spread function involving a given, but not necessarily Toeplitz, matrix A_n by forward projecting and backprojecting a one-pixel point source located at the center of the field-of-view. In matrix terms, the approximate point spread function is just equal to the $\lfloor \frac{n}{2} \rfloor$th column of A_n. The circulant matrix thus obtained was used in [11] as a preconditioner in the steepest descent method to speed up the convergence rate. In this paper, we analyze and apply this circulant matrix as a preconditioner for the conjugate gradient method.

3. Applications and Convergence Analysis

In this section, we study the convergence of our preconditioner for general *deconvolution problems* and *Toeplitz least squares problems*. For such problems, one seeks the solution of a least squares problem

$$\min_x \|b - Ax\|_2, \tag{3.5}$$

in which A is either a rectangular column circulant matrix or a rectangular Toeplitz matrix. These Toeplitz least squares problems occur in a variety of applications especially in signal processing [13,21] and image processing for the 2–D block Toeplitz case [6,10,17,19].

3.1. Deconvolution Problems

We start with the 1-dimensional deconvolution problem. Consider the convolution of a 1–dimensional discrete signal x of length n with a convolution vector h of the form

$$h = [h_{-m+1}, h_{-m+2}, \cdots, h_0, \cdots, h_{m-2}, h_{m-1}]^T.$$

The resulting vector b is of length $2m + n - 2$, and the convolution operation can be expressed in matrix notation as $b = H_{m,n}x$, where $H_{m,n}$ is a column circulant matrix of the form

$$H_{m,n} = \begin{pmatrix} h_{-m+1} & & & 0 \\ h_{-m+2} & h_{-m+1} & & \\ \vdots & \ddots & \ddots & \\ h_0 & & \ddots & h_{-m+1} \\ \vdots & \ddots & & \vdots \\ h_{m-2} & & \ddots & \\ h_{m-1} & \ddots & & h_0 \\ & h_{m-1} & \ddots & \\ & & \ddots & \vdots \\ 0 & & & h_{m-1} \end{pmatrix}. \tag{3.6}$$

In applications such as signal restoration, the observed signal b and the *discrete point spread function* h (essentially any column of $H_{m,n}$) are known, or can be estimated. The aim is to compute x. This is known as discrete deconvolution. Continuous deconvolution can be modeled as an integral equation of the first kind (an ill-posed inverse problem [15]). It is well-known that deconvolution algorithms can be extremely sensitive to noise [15,17]. Since any realistic signal processing problem involves noise, there is a need to incorporate some form of regularization to stabilize the computations. The Tikhonov regularization is probably the best known method for regularizing ill-posed problems [15]. In matrix terms, Tikhonov regularization often amounts to solving the least squares problem

$$\min \left\| \begin{pmatrix} b \\ 0 \end{pmatrix} - \begin{pmatrix} H_{m,n} \\ \mu I_n \end{pmatrix} x \right\|_2, \tag{3.7}$$

where μ is the regularization parameter that depends on the noise level and I_n is the identity matrix. The solution x of (3.7) can be obtained by solving the normal equations

$$(\mu^2 I_n + H_{m,n}^* H_{m,n})x = H_{m,n}^* b. \tag{3.8}$$

We employ the preconditioned conjugate gradient method with generalized Strang circulant preconditioner defined in §2 to solve the normal equations (3.8), and study the convergence rate of the method. Since $H_{m,n}$ is a column circulant matrix, the normal equations matrix $\mu^2 I_n + H_{m,n}^* H_{m,n}$ is a Toeplitz matrix. Hence its generalized Strang preconditioner S_n is the same as the Strang preconditioner.

To consider convergence of our general scheme, we assume that the sequence $\{h_j\}_{j=-\infty}^{\infty}$ is absolutely summable, i.e.

$$\sum_{j=-\infty}^{\infty} |h_j| \leq M < \infty. \tag{3.9}$$

Associate with the sequence $\{h_j\}_{j=-\infty}^{\infty}$ the function g defined by

$$g(\theta) \equiv \sum_{j=-\infty}^{\infty} h_j e^{-ij\theta}, \quad \forall \theta \in [0, 2\pi].$$

Note that g is a function in the Wiener class . Hence $g\overline{g}$ is also a function in the Wiener class and is given by

$$g\overline{g}(\theta) = |g(\theta)|^2 = \sum_{j=-\infty}^{\infty} t_j e^{-ij\theta},$$

where

$$t_j = \sum_{k=-\infty}^{\infty} \overline{h_k} h_{k+j}, \quad j = 0, 1, 2, \ldots. \tag{3.10}$$

Here $\overline{t_j} = t_{-j}$. Let us denote the n-by-n Toeplitz matrix generated by $g\overline{g}$ by T_n (i.e. T_n has diagonals $\{t_j\}_{j=-n+1}^{n-1}$), and the corresponding Strang preconditioner of $\mu^2 I_n + T_n$ by W_n. Since the generating function of $\mu^2 I_n + T_n$ is $\mu^2 + |g(\theta)|^2$ which is positive and in the Wiener class, we can state a clustering results for the preconditioner W_n.

Lemma 3.1. (Chan [2, Corollary 1]) *For any given $\epsilon > 0$, there exist positive integers N_1 and N_2, such that for all $n > N_1$, one can write $\mu^2 I_n + T_n - W_n = R_n + L_n$, with $||R_n||_2 \leq \epsilon$ and rank $L_n \leq N_2$. Moreover, the smallest eigenvalue of W_n is uniformly bounded away from zero. In particular, the spectrum of the preconditioned matrix $W_n^{-1}(\mu^2 I_n + T_n)$ is clustered around 1.*

Now we are going to establish the same result for S_n, when it is used to precondition $\mu^2 I_n + H_{m,n}^* H_{m,n}$. We note by direct verification that the diagonals d_j of $H_{m,n}^* H_{m,n}$ are given by

$$d_j = \sum_{k=-m+1}^{m-1-j} \overline{h_k} h_{k+j}, \quad 0 \leq j \leq n-1 \tag{3.11}$$

with $d_{-j} = \overline{d_j}$. Thus a generating function of $H_{m,n}^* H_{m,n}$ is given by

$$(D_m \star g)(\overline{D_m \star g}),$$

where $D_m \star g$ is the convolution of g with the Dirichlet kernel, i.e.

$$(D_m \star g)(\theta) \equiv \int_0^{2\pi} g(\phi) D_m(\theta - \phi) d\phi = \sum_{k=-m+1}^{m-1} h_k e^{-ik\theta}, \tag{3.12}$$

with

$$D_m(\theta) = \frac{\sin(m - \frac{1}{2})\theta}{\sin(\frac{1}{2}\theta)}.$$

Note that if $h_m \equiv 0$ for $|m| > \beta$, i.e. $H_{m,n}$ has a fixed band-width β, then by (3.12), $D_m \star g = g$ for all $m > \beta$. Hence for all $m > \beta$, $H^*_{m,n}H_{m,n} = T_n$ and $S_n = W_n$. We are now going to show that when the sequence $\{h_j\}_{j=-\infty}^{\infty}$ does not have a fixed band-width, but is absolutely summable, then $H^*_{m,n}H_{m,n}$ and its generalized Strang preconditioner S_n are close to T_n and W_n, respectively, in the 2-norm.

Lemma 3.2. *Let the sequence $\{h_j\}_{j=-\infty}^{\infty}$ be absolutely summable. Then for $m \geq n$,*

$$\lim_{n\to\infty} ||H^*_{m,n}H_{m,n} - T_n||_2 = 0.$$

Proof Note that by Lemma 2.1., for $m \geq n$,

$$\begin{aligned} ||H^*_{m,n}H_{m,n} - T_n||_2 &\leq ||(D_m \star g)\overline{(D_m \star g)} - g\bar{g}||_\infty \\ &= ||(D_m \star g)(\overline{(D_m \star g) - g}) + (D_m \star g - g)\bar{g}||_\infty \\ &\leq ||(D_m \star g)(\overline{(D_n \star g) - g}) + (D_n \star g - g)\bar{g}||_\infty. \end{aligned}$$

Since g is in the Wiener class, $D_n \star g$ converges to g uniformly. Hence the last term in the above equality will tend to 0 as n tends to infinity. ■

Lemma 3.3. *Let the sequence $\{h_j\}_{j=-\infty}^{\infty}$ be absolutely summable. Then for $m \geq n$,*

$$\lim_{n\to\infty} ||W_n - S_n||_2 = 0.$$

Proof Since S_n and W_n are both Strang circulant preconditioners, their eigenvalues are given by

$$\lambda_k(S_n) = \sum_{|j|\leq\lfloor n/2\rfloor} d_j e^{2\pi ijk/n} \quad \text{and} \quad \lambda_k(W_n) = \sum_{|j|\leq\lfloor n/2\rfloor} t_j e^{2\pi ijk/n}, \quad 0 \leq k \leq n-1,$$

see (2.4). Therefore

$$||S_n - W_n||_2 \leq \max_k \left| \sum_{|j|\leq\lfloor n/2\rfloor} (d_j - t_j) e^{2\pi ijk/n} \right| \leq \sum_{|j|\leq\lfloor n/2\rfloor} |d_j - t_j|.$$

However, by (3.10) and (3.11), we have

$$\begin{aligned} ||S_n - W_n||_2 &\leq \sum_{|j|\leq\lfloor n/2\rfloor} \left\{ \sum_{k<-m} |h_k h_{k+j}| + \sum_{k>m} |h_k h_{k-j}| \right\} \\ &\leq \sum_{|k|>m} |h_k| \sum_{j=-\infty}^{\infty} |h_j| \\ &\leq M \sum_{|k|>n} |h_k| \end{aligned}$$

where the last inequality follows from (3.9). As $\{h_k\}_{k=-\infty}^{\infty}$ is absolutely summable, the last summation tends to 0 as n tends to infinity. ■

Combining Lemmas 3.1. to 3.3., we have the following theorem.

Theorem 3.1. *Let the sequence* $\{h_j\}_{j=-\infty}^{\infty}$ *be absolutely summable. Then for any given* $\epsilon > 0$*, there exist positive integers* N_1 *and* N_2*, such that for all* $m \geq n > N_1$*, at most* N_2 *eigenvalues of the matrix* $S_n^{-1}(\mu^2 I_n + H_{m,n}^* H_{m,n}) - I_n$ *have absolute values larger than* ϵ*.*

Proof We note that

$$\mu^2 I_n + H_{m,n}^* H_{m,n} - S_n = \{(\mu^2 I_n + T_n) - W_n\} + \{H_{m,n}^* H_{m,n} - T_n\} + \{W_n - S_n\}.$$

By Lemmas 3.1. to 3.3., we see that $\mu^2 I_n + H_{m,n}^* H_{m,n} - S_n$ can be written as the sum of a low rank matrix and a matrix of small norm for n sufficiently large. Moreover, since the smallest eigenvalue of W_n is also uniformly bounded away from zero and $\lim_{n\to\infty} ||S_n - W_n||_2 = 0$, the smallest eigenvalue of S_n is also uniformly bounded away from zero for sufficiently large n. Thus we see that $S_n^{-1}(\mu^2 I_n + H_{m,n}^* H_{m,n}) - I_n$ can also be written as the sum of a low rank matrix and a matrix of small norm for n sufficiently large. ■

It follows from Theorem 3.1. that the conjugate gradient method, when applied to the preconditioned system $S_n^{-1}(\mu^2 I_n + H_{m,n}^* H_{m,n})$, converges superlinearly (see Chan and Strang [3].)

In practical applications such as signal restoration, the most significant information of regarding the discrete point spread function is often confined to values in the convolution vector h near h_0 [17]. Moreover, the magnitudes of the $h_{|k|}$ decrease significantly as k increases. Figure 1 shows a particular example of a 1–dimensional discrete point spread function in signal restoration with a Gaussian form [17]. When the rate of decrease of $h_{|k|}$ is known, we can get a more precise bound on the convergence rate of our method than (1.2) in terms of the rate of decay of the numbers $h_{|k|}$.

Theorem 3.2. *If*

$$|h_{|k|}| \leq \frac{C}{|k|^{\ell+1}}, \quad k = 0, \pm 1, \pm 2, \ldots, \tag{3.13}$$

for some constants $\ell > 0$ *and* C*, then for sufficiently large* n*, then the error vector* e_q *at the* q*th iteration in exact arithmetic satisfies*

$$|||e_q||| \leq \frac{\tilde{C}^q}{((q-1)!)^{2\ell}} |||e_0|||,$$

where $\tilde{C}$ *is a constant depending on* $\{h_k\}_{j=-\infty}^{\infty}$ *and* ℓ *only and* $|||\cdot|||$ *is defined in (1.3).*

Proof By (3.13), the function g associated with $\{h_j\}_{j=-\infty}^{\infty}$ is a $(\ell+1)$-times differentiable function. It implies that $g\overline{g}$ is also a $(\ell+1)$-times differentiable function. It follows that the rate of decrease of $w_{|j|}$ in (3.10) is given by

$$|w_{|j|}| \leq \frac{\hat{C}}{|j|^{\ell+1}}, \quad j = 0, \pm 1, \pm 2, \ldots.$$

The remaining part of the proof is similar to that in Theorem 3 of Chan [2] and therefore is omitted. ■

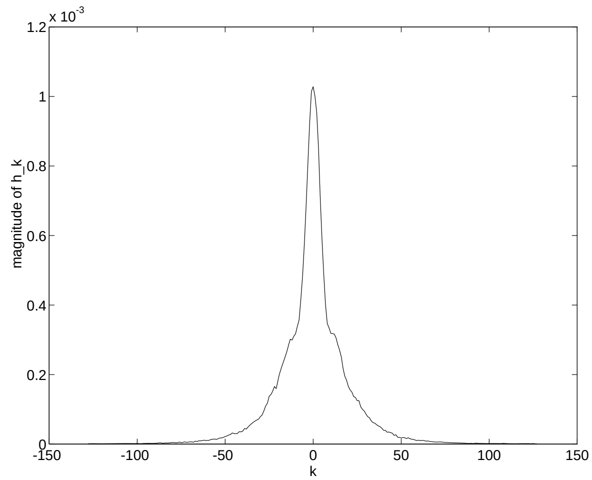

Figure 1. Example of a discrete 1–D Gaussian blur point spread function h

Next we consider 2-dimensional deconvolution problems. In this case, one is still concerned with solving a least squares problem as in (3.5). Here, the matrix A is a block column circulant matrix with column circulant blocks. More precisely,

$$A = \begin{pmatrix} A^{(-m+1)} & & & 0 \\ A^{(-m+2)} & A^{(-m+1)} & & \\ \vdots & \ddots & \ddots & \\ A^{(0)} & & \ddots & A^{(-m+1)} \\ \vdots & \ddots & & \vdots \\ A^{(m-2)} & & \ddots & \\ A^{(m-1)} & \ddots & & A^{(0)} \\ & A^{(m-1)} & \ddots & \\ & & \ddots & \vdots \\ 0 & & & A^{(m-1)} \end{pmatrix} \qquad (3.14)$$

with each subblock $A^{(j)}$ being a $2m+n-2$-by-n matrix of the form given by (3.6). We note that A^*A will be a n-block-by n-block block Toeplitz matrix with n-by-n Toeplitz blocks. The 2-dimensional deconvolution problem has n^2 unknowns since A has n^2 columns.

The generalized Strang circulant preconditioner S for A^*A is related to the level–2 Strang circulant preconditioner proposed by T. Chan and Olkin [10]. For block Toeplitz matrices with Toeplitz blocks that are generated by a fixed generating function, the convergence rate of the method has been discussed in [7, Theorem 3]. Here however, in contrast to the 1–D case, the block Toeplitz matrices with Toeplitz blocks A^*A do not have a fixed generating function – their diagonals vary with n. However, if we assume that the diagonals of A are absolutely summable, i.e., $||A||_1 \leq M < \infty$, then the same arguments used in 1–D case can also give us the same convergence result (cf, [7, Theorem 3]).

Theorem 3.3. *Consider the 2-D deconvolution problem with n^2 unknowns. For A given in (3.14), suppose $||A||_1 \leq M < \infty$. Then for any given $\epsilon > 0$, there exists a positive integer N, such that for all $n > N$, at most $O(n)$ eigenvalues of $S^{-1}(\mu^2 I + A^* A) - I$ have absolute values larger than ϵ.*

We note that the ground-based atmospheric imaging problem considered in §4 satisfies the conditions of Theorem 3.3..

3.2. Toeplitz Least Squares Problems

For simplicity, we first consider *pre-windowed Toeplitz least squares problems* [13]. The general Toeplitz least squares problems will be discussed later. For pre-windowed Toeplitz least squares problems, the Toeplitz matrices $A_{m,n}$ are given by

$$A_{m,n} = \begin{pmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ a_{m-n} & \cdots & \cdots & a_0 \\ \vdots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_{m-1} & \cdots & \cdots & a_{m-n} \end{pmatrix}$$

with $m \geq n$ and full column rank n. The solution of these least squares problem $||b - A_{m,n}x||_2$ can be found by solving the normal equations

$$A^*_{m,n} A_{m,n} x = A_{m,n} b.$$

Since $A_{m,n}$ is not column circulant, the normal equations matrix $A^*_{m,n} A_{m,n}$ is non-Toeplitz. We employ the generalized Strang preconditioner $(S^*_n S_n)^{1/2}$ for the normal equations matrix and solve the preconditioned systems

$$(S^*_n S_n)^{-1/2} A^*_{m,n} A_{m,n} x = (S^*_n S_n)^{-1/2} A_{m,n} b.$$

Here S_n is a circulant matrix with its $\lfloor \frac{n}{2} \rfloor$th column given by $A^*_{m,n} A_{m,n} e_{\lfloor \frac{n}{2} \rfloor}$.

Since $A_{m,n}$ is lower-triangular, we can write

$$A^*_{m,n} A_{m,n} = B^*_{m,n} B_{m,n} - U^*_n U_n, \tag{3.15}$$

where U_n is the n-by-n upper triangular Toeplitz matrix with its first row given by

$$[0, a_{m-1}, a_{m-2}, \cdots, a_{m-n+1}]$$

and $B_{m,n}$ is the $(m+n)$-by-n column circulant matrix obtained by stacking $A_{m,n}$ on top of U_n. By (3.15), we see that S_n is the difference of two circulant matrices,

$$S_n = S^{(1)}_n - S^{(2)}_n \tag{3.16}$$

where the $\lfloor \frac{n}{2} \rfloor$th columns of $S_n^{(1)}$ and $S_n^{(2)}$ are given by

$$B_{m,n}^* B_{m,n} e_{\lfloor \frac{n}{2} \rfloor} \quad \text{and} \quad U_n^* U_n e_{\lfloor \frac{n}{2} \rfloor}$$

respectively. We note that $S_n^{(1)}$ is Hermitian and $S_n^{(2)}$ is in general not.

Since $B_{m,n}$ is column circulant, we can use the results in §3.1 to study the approximation properties of $S_n^{(1)}$. For this, we associate the (j,k)th entries a_{j-k} of $A_{m,n}$ with the function

$$f(\theta) = \sum_{k=0}^{\infty} a_k e^{-ik\theta}$$

defined on $[0, 2\pi]$. We will assume that f is a function in the Wiener class (i.e. the sequence $\{a_k\}_{k=0}^{\infty}$ is absolutely summable) and has no zero on $[0, 2\pi]$. Then using arguments similar to that used in Lemmas 3.2., 3.3. and Theorem 3.1., we have the following Lemma.

Lemma 3.4. *Let f be in the Wiener class. Then for any given $\epsilon > 0$, there exist positive integers N_1 and N_2, such that for all $m \geq n > N_1$, one can write $B_{m,n}^* B_{m,n} - S_n^{(1)} = R_n + L_n$, with $||R_n||_2 \leq \epsilon$ and* rank *$L_n \leq N_2$. If moreover f has no zeros on $[0, 2\pi]$, then the spectra of $S_n^{(1)}$ and of its inverse are uniformly bounded for all large n.*

Thus for large n, $B_{m,n}^* B_{m,n} - S_n^{(1)}$ is the sum of a small norm matrix and a low rank matrix. Next we show that U_n is a sum of a low rank matrix and small norm matrix and $S_n^{(2)}$ is also small norm matrix for large n.

Lemma 3.5. *Let f be in the Wiener class. Then for any given $\epsilon > 0$, there exists integer $N_1 > 0$, such that for $n > N_1$,*

$$U_n^* U_n = R_n + L_n,$$

with $||R_n||_2 \leq \epsilon$ and rank $L_n \leq 2N_1$. Moreover, we have

$$\lim_{n \to \infty} ||S_n^{(2)}||_2 = 0.$$

Proof Since $\{a_k\}_{k=0}^{\infty}$ is absolutely summable, we have

$$\sum_{j=0}^{\infty} |a_j| \leq M < \infty,$$

and also for any given $\epsilon > 0$, there exists a positive integer N_1 such that

$$\sum_{k=N_1}^{\infty} |a_k| \leq \frac{\epsilon}{M}.$$

For $n > N_1$, we partition U_n as $\tilde{L}_n + \tilde{R}_n$, where the first $(n - N_1)$ columns of $\tilde{L}_n$ are zero vectors and the last N_1 columns of $\tilde{L}_n$ are given by the last N_1 columns of U_n. Clearly $\tilde{L}_n$ is matrix of rank N_1 and

$$||\tilde{R}_n||_1 \leq \sum_{j=m-n+1+N_1}^{m-1} |a_j| \leq \sum_{j=N_1+1}^{m-1} |a_j| \leq \frac{\epsilon}{M}.$$

Thus

$$U_n^* U_n = (\tilde{L}_n + \tilde{R}_n)^*(\tilde{L}_n + \tilde{R}_n) = L_n + R_n,$$

where

$$\text{rank } L_n = \text{ rank } (\tilde{L}_n^* \tilde{R}_n + \tilde{R}_n^* \tilde{L}_n + \tilde{L}_n^* \tilde{L}_n) \leq 2N_1$$

and

$$\|R_n\|_2 \leq \|\tilde{R}_n^* \tilde{R}_n\|_2 \leq \frac{\epsilon}{M}.$$

Similarly, for $n > 2N_1$, we get

$$\begin{aligned}
\|S_n^{(2)}\|_2^2 &\leq \|S_n^{(2)}\|_1 \|S_n^{(2)}\|_\infty \\
&\leq \|U_n^* U_n e_{\frac{n}{2}}\|_1^2 \\
&\leq \|U_n\|_1^2 \|U_n e_{\frac{n}{2}}\|_1^2 \\
&\leq M \left(\sum_{k=m-\lfloor \frac{n}{2} \rfloor + 1}^{m-1} |a_k| \right)^2 \\
&\leq \epsilon^2.
\end{aligned}$$

■

Using the above two lemmas, we can show that $(S_n^* S_n)^{1/2}$ is close to $S_n^{(1)}$.

Lemma 3.6. *Let f be a function in the Wiener class with no zeros on $[0, 2\pi]$. Then*

$$\lim_{n\to\infty} \|(S_n^* S_n)^{1/2} - S_n^{(1)}\|_2 = 0.$$

In particular, the spectra of $(S_n^* S_n)^{1/2}$ and of its inverse are uniformly bounded.

Proof We first note that S_n, $S_n^{(1)}$, $S_n^{(2)}$ and $(S_n^* S_n)^{1/2}$ are all circulant matrices and hence can be diagonalized by the same Fourier matrix. For simplicity, let us denote their eigenvalues by λ_k, $\lambda_k^{(1)}$, $\lambda_k^{(2)}$ and $\tilde{\lambda}_k$ respectively. By (3.16), λ_k can be expressed as

$$\lambda_k = \lambda_k^{(1)} - \lambda_k^{(2)}, \quad 0 \leq k \leq n-1.$$

Hence

$$\tilde{\lambda}_k^2 - (\lambda_k^{(1)})^2 = \lambda_k^* \lambda_k - (\lambda_k^{(1)})^2 = |\lambda_k^{(2)}|^2 - 2\lambda_k^{(1)} \text{Re}(\lambda_k^{(2)})$$

where $\text{Re}(\cdot)$ denotes the real part of a complex number. By Lemmas 3.4. and 3.5., we know that as n tends to infinity, $\lambda_k^{(1)}$ are uniformly bounded while $|\lambda_k^{(2)}|$ will tend to zero uniformly. In particular, we have

$$\lim_{n\to\infty} \max_{0\leq k\leq n-1} |\tilde{\lambda}_k^2 - (\lambda_k^{(1)})^2| = \lim_{n\to\infty} \max_{0\leq k\leq n-1} \left| |\lambda_k^{(2)}|^2 - 2\lambda_k^{(1)} \text{Re}(\lambda_k^{(2)}) \right| = 0.$$

Since $\lambda_k^{(1)}$ are uniformly bounded, this implies that $\tilde{\lambda}_k$ are also uniformly bounded and hence

$$\lim_{n\to\infty} \max_{0\leq k\leq n-1} |\tilde{\lambda}_k - \lambda_k^{(1)}| = 0.$$

■

Combining the last three lemmas, we can easily prove that the preconditioned matrices have clustered spectra.

Theorem 3.4. *Let f be a function in the Wiener class and has no zeros on $[0, 2\pi]$. Then for any given $\epsilon > 0$, there exist positive integers N_1 and N_2 such that for all $n > N_1$, at most N_2 eigenvalues of $(S_n^* S_n)^{-1/2} A_{m,n}^* A_{m,n}$ have absolute values larger than ϵ.*

Proof We note that

$$(S_n^* S_n)^{1/2} - A_{m,n}^* A_{m,n} = \{(S_n^* S_n)^{1/2} - S_n^{(1)}\} + \{S_n^{(1)} - B_{m,n}^* B_{m,n}\} + U_n^* U_n.$$

Combining the results in Lemmas 3.4., 3.5. and 3.6., we thus see that $(S_n^* S_n)^{1/2} - A_{m,n}^* A_{m,n}$ can be written in a sum of a small norm matrix and a low rank matrix. The result now follows by noting that $(S_n^* S_n)^{-1/2}$ is uniformly invertible and

$$(S_n^* S_n)^{-1/2} A_{m,n}^* A_{m,n} = I_n + (S_n^* S_n)^{-1/2} \{A_{m,n}^* A_{m,n} - (S_n^* S_n)^{1/2}\}.$$

■

From Theorem 3.4. we have the desired clustering result. It can also be shown, in a manner similar to the derivation in §4 of [6], that if the condition number of $A_{m,n}$ is of $O(n^\gamma)$, $\gamma > 0$, then the conjugate gradient method converges in at most $O(\gamma \log n + 1)$ steps. Since each iteration requires $O(m \log n)$ operations using the FFT, it follows that the total complexity of the algorithm is only $O(\gamma m \log^2 n + m \log n)$. When $\gamma = 0$, i.e., $\kappa(A_{m,n}) = O(1)$, the number of iterations required for convergence is of $O(1)$. Hence the complexity of the algorithm reduces to $O(m \log n)$, for sufficiently large n. In contrast, the method converges just linearly for the non-preconditioned case, as is illustrated by numerical examples in the next section.

Finally we consider the general Toeplitz least squares problems. In this case, the rectangular Toeplitz matrices $A_{m,n}$ are given by

$$A_{m,n} = \begin{pmatrix} a_0 & a_{-1} & \cdots & a_{-n+1} \\ a_1 & a_0 & \ddots & \vdots \\ \vdots & & \ddots & a_{-1} \\ a_{m-n} & \cdots & \cdots & a_0 \\ \vdots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_{m-1} & \cdots & \cdots & a_{m-n} \end{pmatrix}.$$

We remark that in the pre-windowed case the entries $[A_{m,n}]_{j,k}$ for $j < k$ are assumed to be zero. Similar to (3.15), we have

$$A_{m,n}^* A_{m,n} = B_{m,n}^* B_{m,n} - U_n^* U_n - V_n^* V_n,$$

where V_n is the n-by-n lower triangular Toeplitz matrix with its first column given by

$$[0, a_{-n+1}, a_{-n+2}, \cdots, a_{-1}]^T.$$

and $B_{m,n}$ is the $(m + 2n)$-by-n column circulant matrix given by

$$B_{m,n} = \begin{pmatrix} V_n \\ A_{m,n} \\ U_n \end{pmatrix}.$$

Thus instead of (3.16), we have

$$S_n = S_n^{(1)} - S_n^{(2)} - S_n^{(3)}$$

where the $\lfloor \frac{n}{2} \rfloor$th columns of $S_n^{(1)}$, $S_n^{(2)}$ and $S_n^{(3)}$ are given by

$$B_{m,n}^* B_{m,n} e_{\lfloor \frac{n}{2} \rfloor}, \quad U_n^* U_n e_{\lfloor \frac{n}{2} \rfloor} \quad \text{and} \quad V_n^* V_n e_{\lfloor \frac{n}{2} \rfloor}$$

respectively. In order to prove convergence, we assume similar to the pre-windowed case that the function

$$f(\theta) = \sum_{k=-\infty}^{\infty} a_k e^{-ik\theta}$$

defined on $[0, 2\pi]$ is in the Wiener class and has no zero on $[0, 2\pi]$. Then by using similar arguments as in Lemma 3.5., we can prove that for sufficiently large n, $V_n^* V_n$ is a sum of a low rank matrix and a small norm matrix, and that $\|S_n^{(3)}\|_2$ is small. Therefore, the conclusion of Lemma 3.6. still holds and hence the preconditioned matrices $(S_n^* S_n)^{-1/2} A_{m,n}^* A_{m,n}$ will still have clustered spectra around 1. Accordingly, the preconditioned conjugate gradient method with the generalized Strang preconditioner will also be an efficient algorithm for solving Toeplitz least squares equations.

4. Numerical Results

In this section the effectiveness of our preconditioner is illustrated by some numerical examples. For each of the 1–D examples we use the vector of all ones as the right hand side and the zero vector as the initial guess. The stopping criteria is $||u^{(j)}||_2/||u^{(0)}||_2 < 10^{-7}$, where $u^{(j)}$ is the normal equations residual after j iterations. We conclude with a 2–D problem arising in ground–base astronomical imaging. All computations were performed using Matlab. In the first two test examples, we consider column circulant matrices $H_{m,n}$ as in (3.6) with entries given by $\{h_k\}_{k=-\infty}^{\infty}$ that satisfy the conditions of Theorem 3.1.. We note that in Example 2, the bandwidth of the column circulant matrices is set to 63.

Example 1. $h_k = 1/(k+1)^\alpha, \quad k = 0, \pm 1, \pm 2, \ldots.$
Example 2. $h_k = 1/(k+1)^\alpha, \quad k = 0, \pm 1, \pm 2, \ldots \pm 31.$

In Table 1, we present the number of iterations needed for convergence when using the preconditioned conjugate gradient method with no preconditioner, our generalized Strang preconditioner, the displacement preconditioner as defined in [8], and the preconditioner based on partitioning rectangular Toeplitz matrices as defined in [6]. We denote these by "no", "gens", "disp" and "part" respectively.

Next, in Examples 3–5, we consider matrices $A_{m,n}$ that are not column circulant but Toeplitz. Hence the normal equations matrices are not Toeplitz, and therefore the original Strang preconditioner is not defined. Here we will use the generalized Strang preconditioner and compare it with the other circulant preconditioners. We denote the entries of the first columns and the first rows of the matrices $A_{m,n}$ by $c(\cdot)$ and $r(\cdot)$ respectively. We remark that the matrices in Examples 3 and 4 are pre-windowed Toeplitz matrices while the matrices in Example 5 are general Toeplitz matrices. The convergence results are listed in Tables 2 and 3.

Table 1. Numbers of iterations for Examples 1 and 2 when $m = n$

Example 1	$\alpha = 2$				$\alpha = 1.1$			
n	no	gens	disp	part	no	gens	disp	part
17	9	5	5	5	10	7	5	6
33	13	4	5	5	17	6	5	5
65	16	4	5	5	28	6	5	5
129	20	4	5	5	42	5	5	5
257	22	5	5	5	57	5	5	5
Example 2	$\alpha = 2$				$\alpha = 1.1$			
n	no	gens	disp	part	no	gens	disp	part
65	17	5	5	5	28	7	6	7
129	20	5	5	5	43	6	6	6
257	22	5	5	5	56	6	6	6

Table 2. Numbers of iterations for Examples 3, 4 and 5 when $m = n$.

	Example 3				**Example 4**				**Example 5**			
n	no	gens	disp	part	no	gens	disp	part	no	gens	disp	part
17	15	6	6	6	12	7	6	6	14	9	10	8
33	22	6	6	6	17	7	6	7	29	6	14	10
65	30	6	7	6	21	7	7	7	56	6	11	9
129	31	6	7	7	25	7	7	7	98	6	9	8
257	31	6	7	7	30	7	7	7	142	6	8	7

Example 3.
$$\begin{aligned} c(k) &= e^{-0.1k^2}, \qquad k = 1, \ldots, m \\ r(1) &= c(1), \\ r(k) &= 0, \qquad k = 2, \ldots, n. \end{aligned}$$

Example 4.
$$\begin{aligned} c(k) &= 1/k^{1.1}, \qquad k = 1, \ldots, m \\ r(1) &= c(1), \\ r(k) &= 0, \qquad k = 2, \ldots, n. \end{aligned}$$

Example 5.
$$\begin{aligned} c(k) &= e^{-0.1k^2}, \qquad k = 1, \ldots, m \\ r(k) &= e^{-0.1k^2}, \qquad k = 1, \ldots, n. \end{aligned}$$

From Tables 1, 2 and 3, we observe that the number of iterations needed for convergence for the preconditioned systems is essentially independent of the sizes of the matrices. These numerical results illustrate that the generalized Strang circulant

Table 3. Numbers of iterations for Examples 3, 4 and 5 when $m = 2n$.

	Example 3				**Example 4**				**Example 5**			
n	no	gens	disp	part	no	gens	disp	part	no	gens	disp	part
17	9	4	6	5	12	7	6	6	24	11	16	12
33	15	4	6	5	17	7	6	7	46	9	17	11
65	22	4	5	5	21	7	7	7	85	9	14	10
129	30	4	5	5	25	7	7	7	137	9	12	9
257	31	4	5	4	30	7	7	7	186	9	10	9

preconditioner can significantly reduce the number of iterations needed for convergence. The performance of all three circulant preconditioners is similar for these 1–D problems. We stress, however, that the generalized Strang circulant preconditioner can be defined for more general matrices such as the non–Toeplitz normal equation matrices arising from Toeplitz least square problems.

Finally we consider a 2-dimensional deconvolution problem arising in ground-based atmospheric imaging. We use the preconditioned conjugate gradient algorithm with the generalized Strang circulant preconditioner to remove the blurring in an image resulting from the effects of atmospheric turbulence. The problem consists of a 256-by-256 image of an ocean reconnaissance satellite observed by a simulated ground-based imaging system together with a 256-by-256 image of a guide star observed under similar circumstances (see Figure 2.) The data are provided by the Phillips Air Force Laboratory at Kirkland AFB, NM [1]. The imaging system detects the atmospheric distortions using a natural guide star image. A wavefront sensor measures the optical distortions which can then be digitized into a blurred image of the guide star pixel. To form the discrete point spread function h, the rows of the blurred pixel image are stacked into a column vector. Then the point spread function matrix A is given in block form as in (3.14) with h as its first column. Moreover A satisfies the conditions of Theorem 3.3., since the guide star for the atmospheric imaging problem yields a Gaussian point spread function [20].

In Figures 3 and 4, we present restorations without and with using the generalized Strang preconditioner described in §3. The regularization parameter μ in both cases is chosen to be 0.01. From the figures, we observe that when no preconditioner is used, an acceptable restoration is achieved after 34 iterations. Essentially, the same restoration is achieved in 3 iterations when preconditioning is used. We remark that the cost per iteration using Strang's preconditioner is less than that using the preconditioner proposed in [20]. This is because we use circulant based preconditioning, whereas an inverse Toeplitz based preconditioner is used in [20], which doubles the dimension of the problem being solved. In particular roughly 0.61×10^8 floating point operations per iteration are used for our circulant based deconvolution, while roughly 0.98×10^8 per iteration are necessary using the method in [20]. The count for no preconditioning is 0.50×10^8.

For comparison, we also used T. Chan's circulant preconditioner [9] to test the restoration of the above atmospheric image. We report that about the same restoration is achieved in 6 iterations when T. Chan's circulant preconditioner is used. This is twice the number of iterations required using our generalized Strang preconditioner, and both schemes require the same number of operations per iteration. Figure 5 shows the 2-norm of the normal equations residuals of these preconditioned conjugate gradient methods. We observe that the decrease of residuals when Strang's preconditioning is used is faster than that when T. Chan's preconditioning is used.

In summary, these preliminary experiments suggest that the preconditioned conjugate gradient algorithm with the generalized Strang circulant preconditioner may be an efficient and effective method for deconvolution problems.

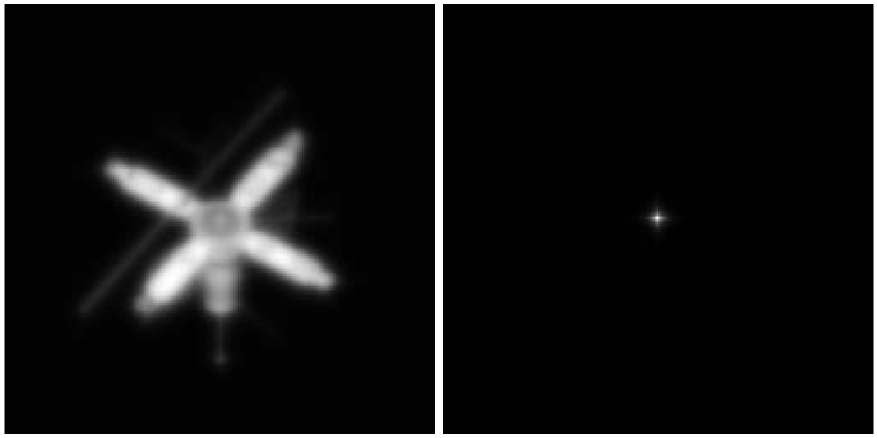

Figure 2. Observed Image (left) and guide star image (right).

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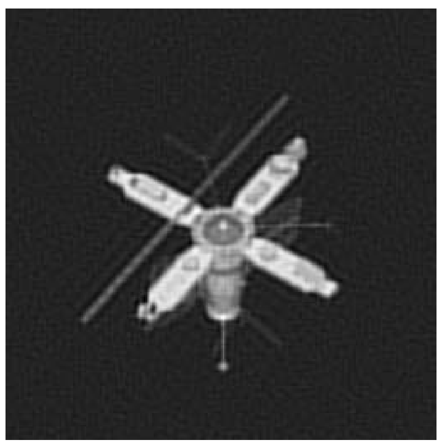
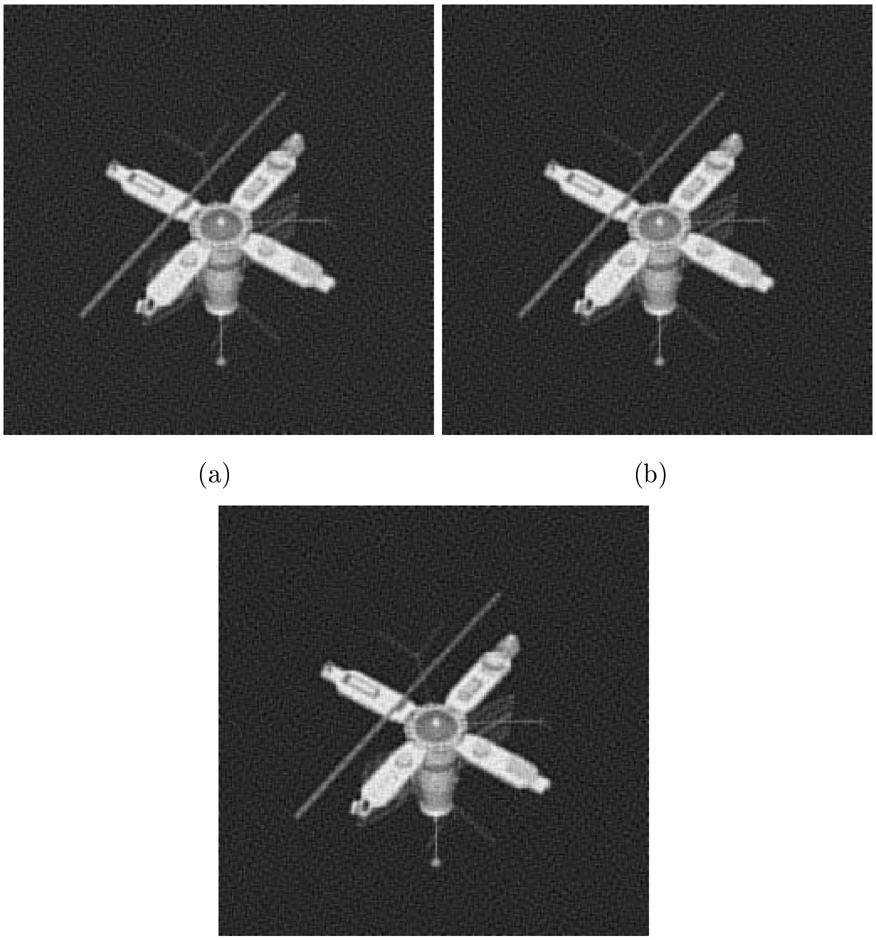
(a) (b)



(c)

Figure 3. Restored images with no preconditioning: (a) 3 iterations, (b) 15 iterations and (c) 34 iterations respectively.



(a) (b) (c)

Figure 4. Restored images using the generalized Strang preconditioner: (a) 1 iteration, (b) 3 iterations and (c) using T. Chan's preconditioner: 6 iterations.

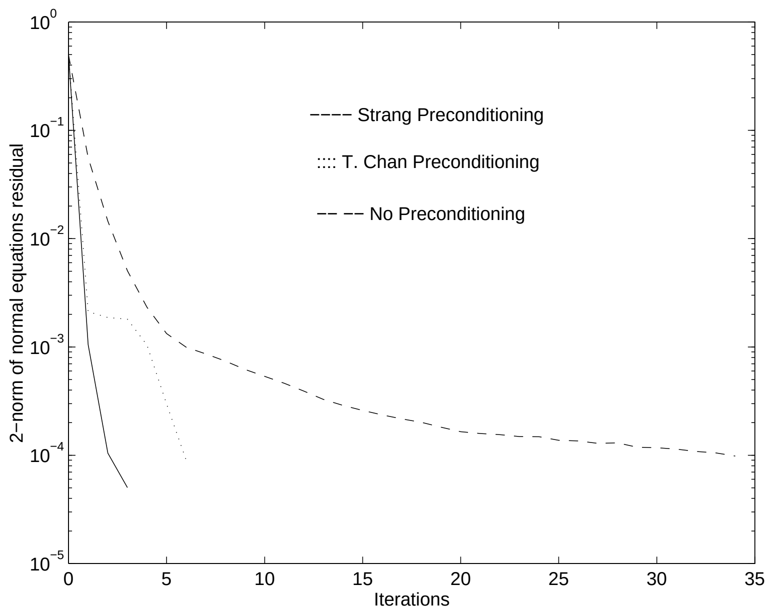


Figure 5. Normal equations residuals for using Strang's, T. Chan's and without using preconditioners.

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