The Numerical Solution of the Biharmonic Equation by Conformal Mapping

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Abstract. The solution to the biharmonic equation in a simply connected region Ω in the plane is computed in terms of the Goursat functions. The boundary conditions are conformally transplanted to the disk with a numerical conformal map. A linear system is obtained for the Taylor coefficients of the Goursat functions. The coefficient matrix of the linear system can be put in the form $I + K$ where K is the discretization of a compact operator. K can be thought of as the composition of a block Hankel matrix with a diagonal matrix. The compactness leads to clustering of eigenvalues and the Hankel structure yields a matrix-vector multiplication cost of $O(N \log N)$. Thus if the conjugate gradient method is applied to the system then superlinear convergence will be obtained. Numerical results are given to illustrate the spectrum clustering and superlinear convergence.

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1. Introduction. Boundary value problems for the biharmonic equation in two dimensions arise in the computation of the Airy stress function for plane stress problems [KK], [Mik], [Musk], and in steady Stokes flow of highly viscous fluids [MT, Chap. 22], [Poz]. Integral equations methods are a popular choice for the numerical solution of these equations [GGMa], [MG], [K, and references there], [Poz]. The application of conformal mapping to this problem, though classical, is less well known [KK], [Musk]. Unlike the Laplace equation, the biharmonic equation is not preserved under conformal transplantation. However, a biharmonic function and its boundary values can be represented in terms of the analytic Goursat functions and this representation can be transplanted with a conformal map to a computational region, such as a disk, an ellipse, or an annulus, where the boundary value problem can be solved more easily.

In this paper, we consider simply-connected regions with analytic boundaries and use the unit disk as our computational region. In our examples, the conformal map f from the unit disk to the region is either known explictly or approximated numerically. The

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boundary conditions for the biharmonic function are then transplanted by f to the disk and a linear system for the Taylor coefficients of the Goursat functions in the disk is obtained and solved efficiently by conjugate-gradient-like methods. If the boundary of the target region is smooth enough (analytic in our examples), the continuous problem can be posed as a compact operator acting on some appropriate Banach space. This will lead to a clustering of the spectrum and hence superlinear convergence.

We expect to be able to generalize this work to cases where the conformal map f is a Faber series map from an ellipse, cross-shaped or spoke-like region as in [DE] and [DEP]. If the target region has elongated sections, the conformal map from the disk may be severely ill-conditioned and an ellipse or cross-shaped region may provide a better computational region. In [GGMa] the Sherman-Lauricella equation is solved for spoke-like regions which provide difficult regions for plane stress and plane strain problems. We anticipate that our Faber series methods may have advantages for such highly distorted regions. In cases for which the target region is not too distorted, so that the map from the disk is not too severely ill-conditioned, our method may also have some advantages. For instance, if several boundary value problems have to be solved for the same region, so that the conformal map only has to be computed once, our method, which is based on the fast Fourier transform (FFT), will give accurate answers in $O(N \log N)$ for moderate sized N. The methods in [GGMa] use the fast multipole method, which costs only $O(N)$, but with a large constant, so that large N are required in practice for it to be faster than the FFT. Below, we will use the FFT-based numerical conformal mapping method given in [Weg]. Introductions to numerical conformal mapping can be found in [Ga] and [He].

The outline of the paper is as follows. In section 2, we discuss the solution of boundary value problems for the biharmonic equation in terms of Goursat functions and the conformal map from the disk to the plane region. In section 3, we discuss the special structure of the exact linear system. We will see that the coefficient matrix of the (infinite) linear system is of the form $I+HD$ where I is the identity matrix, D a diagonal matrix, and H is a block Hankel matrix. (A *Hankel matrix* is constant on the antidiagonals.) It will be seen that HD actually can be represented as a compact operator with a one dimensional null space. This system can be symmetrized and solved (up to the null vector) using the conjugate gradient method. In section 4, we formulate the discrete problem. We will show how the conjugate gradient method is applied to the discrete system and how the matrix-vector multiplication can be carried out in $O(N \log N)$. In section 5, we give several numerical examples which illustrate the spectrum clustering, the superlinear convergence, and the discretization error.

2. The biharmonic equation. Here we will follow the presentation in [KK] and [Musk]. We wish to find the Airy stress function u for a simply connected region Ω with a smooth boundary Γ in the ζ-plane. Then u satisfies the biharmonic equation,

$$\Delta^2 u = 0$$

for $\zeta = \eta + i\mu \in \Omega$. The two fundamental boundary value problems in elasticity seek to find u given the external stresses or external displacements on the boundary Γ. Both of these problems amount to specifying

$$u_\eta = G_1 \quad \text{and} \quad u_\mu = G_2$$

on Γ. The function u can be represented as

$$u(\zeta) = Re(\overline{\zeta}\phi(\zeta) + \chi(\zeta)),$$

where $\phi(\zeta)$ and $\chi(\zeta)$ are analytic functions in Ω known as the *Goursat functions.* Letting $G = G_1 + iG_2$, the boundary conditions for the first fundamental problem become

$$\phi(\zeta) + \zeta\overline{\phi'(\zeta)} + \overline{\psi(\zeta)} = G(\zeta), \quad \zeta \in \Gamma \tag{1}$$

where $\psi(\zeta) = \chi'(\zeta)$. The second fundamental problem leads to similar conditions. For simplicity, in this paper, we will only concentrate on the first boundary conditions (1).

We remark that $\phi(\zeta)$ and $\psi(\zeta)$ are not unique. In fact, if $\phi(\zeta)$ and $\psi(\zeta)$ represent any solution of the problem, then so does $\phi(\zeta) + Ci\zeta + \gamma$ and $\psi(\zeta) + \gamma'$, where $C \in \mathbf{R}$, $\gamma \in \mathbf{C}$, and $\gamma' \in \mathbf{C}$. Thus, the constants C and γ must be specified for uniqueness of ϕ. These constants are determined below.

The problem at this point is to find ϕ and ψ analytic in Ω and satifying (1). One approach is to represent ϕ and ψ as Cauchy-type integrals of a density function on Γ. This leads to the *Sherman-Lauricella equation,* a Fredholm integral equation for the density function which can be solved efficiently by the fast multipole method [GGMa]. In this paper, we propose to solve it by using numerical conformal mapping coupled with the conjugate gradient method.

Let $\zeta = f(z)$ be the conformal map from the unit disk to Ω, fixing $f(0) = 0 \in \Omega$. Then with $d(z) := f(z)/\overline{f'(z)}$, $\phi(z) := \phi(f(z)), \psi(z) := \psi(f(z))$, and $G(z) := G(f(z))$, equation (1) transplants to the disk as

$$\phi(z) + d(z)\overline{\phi'(z)} + \overline{\psi(z)} = G(z), \quad |z| = 1. \tag{2}$$

Let

$$\phi(z) = \sum_{k=1}^{\infty} a_k z^k \quad \text{and} \quad \psi(z) = \sum_{k=0}^{\infty} b_k z^k.$$

Notice that the sum for ϕ begins at $k = 1$. This fixes the constant γ mentioned above for uniqueness by requiring $\phi(0) = a_0 = 0$. After transplanting to the disk, the other constant is determined by setting $Im(a_1/f'(0)) = 0$.

The problem is to find the a_k's and the b_k's. For $|z| = 1$, define the Fourier series

$$d(z) := f(z)/\overline{f'(z)} = \sum_{k=-\infty}^{\infty} h_k z^k, \quad G(z) = \sum_{k=-\infty}^{\infty} A_k z^k.$$

Substituting into (2) gives a linear system of equations for the a_k's and b_k's,

$$a_j + \sum_{k=1}^{\infty} k\overline{a}_k h_{k+j-1} = A_j, \quad j = 1, 2, 3, \ldots \tag{3}$$

$$\overline{b}_j + \sum_{k=1}^{\infty} k\overline{a}_k h_{k-j-1} = A_{-j}, \quad j = 0, 1, 2, \ldots \tag{4}$$

If (3) is solved for the a_k's, then the b_k's can be easily computed from (4). Thus, in this paper, we will concentrate on an efficient method for solving (3).

There is also a moment condition to be satisfied by the data. After transplantation to the disk, this condition can be stated as $Re[\int_{|z|=1} G(z)\overline{f'(z)d\overline{z}}] = 0$. This moment condition will assure the existence of a solution. Our assumption is that all data studied in this paper satisfy this equation.

Before proceeding, it should be noted that if our boundary data corresponds to $G = 0$ then the only possible (nonzero) choice for ϕ is $\phi(z) = Cif(z)$, for some nonzero $C \in \mathbf{R}$. This implies that the null space corresponding to the infinite system in (3) is one dimensional and the eigenvector spanning this space is given by $a_k = ic_k, k = 1, 2, 3, \cdots$ where $f(z) = \sum_{k=1}^{\infty} c_k z^k$.

3. Compact Operators. Taking real and imaginary parts of equation (3) gives us

$$\alpha_j + \sum_{k=1}^{\infty} k(\eta_{k+j-1}\alpha_k + \gamma_{k+j-1}\beta_k) = B_j, \quad j = 1, 2, 3, \ldots \tag{5}$$

$$\beta_j + \sum_{k=1}^{\infty} k(\gamma_{k+j-1}\alpha_k - \eta_{k+j-1}\beta_k) = C_j, \quad j = 1, 2, 3, \ldots \tag{6}$$

where we have used the notation $a_k = \alpha_k + i\beta_k$, $h_k = \eta_k + i\gamma_k$, and $A_k = B_k + iC_k$. For visualization purposes, we combine equations (5) and (6) into a doubly infinite matrix equation in which the two sums are combined into a block Hankel matrix composed with a diagonal matrix. In fact, (5) and (6) can be written as

$$(I_\infty + H_{r,\infty}D_\infty)\underline{\alpha} + H_{i,\infty}D_\infty\underline{\beta} = \underline{B} \tag{7}$$

$$(I_\infty - H_{r,\infty}D_\infty)\underline{\beta} + H_{i,\infty}D_\infty\underline{\alpha} = \underline{C} \tag{8}$$

so that

$$\left(\begin{pmatrix} I_\infty & 0 \\ 0 & I_\infty \end{pmatrix} + \begin{pmatrix} H_{r,\infty} & H_{i,\infty} \\ H_{i,\infty} & -H_{r,\infty} \end{pmatrix}\begin{pmatrix} D_\infty & 0 \\ 0 & D_\infty \end{pmatrix}\right)\begin{pmatrix} \underline{\alpha} \\ \underline{\beta} \end{pmatrix} = \begin{pmatrix} \underline{B} \\ \underline{C} \end{pmatrix} \tag{9}$$

where $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots)^T$, $\underline{\beta} = (\beta_1, \beta_2, \ldots)^T$, $\underline{B} = (B_1, B_2, \ldots)^T$, $\underline{C} = (C_1, C_2, \ldots)^T$, I_∞ is the infinite identity matrix, $D_\infty = diag(1, 2, \ldots)$, $H_{r,\infty}$ is an infinite Hankel matrix generated by the η_k, and $H_{i,\infty}$ is an infinite Hankel matrix generated by the γ_k.

Now suppose $(\underline{\alpha}, \underline{\beta})$ represents a solution to (5),(6). Define

$$\underline{x} = \begin{pmatrix} D_\infty^{1/2}\underline{\alpha} \\ D_\infty^{1/2}\underline{\beta} \end{pmatrix}, \quad \underline{r} = \begin{pmatrix} D_\infty^{1/2}\underline{B} \\ D_\infty^{1/2}\underline{C} \end{pmatrix}.$$

Then (9) can be written as

$$(I_\infty + M_\infty)\,\underline{x} = \underline{r} \tag{10}$$

where M_∞ is given by

$$M_\infty = \begin{pmatrix} M_{r,\infty} & M_{i,\infty} \\ M_{i,\infty} & -M_{r,\infty} \end{pmatrix} = \begin{pmatrix} D_\infty^{1/2} H_{r,\infty} D_\infty^{1/2} & D_\infty^{1/2} H_{i,\infty} D_\infty^{1/2} \\ D_\infty^{1/2} H_{i,\infty} D_\infty^{1/2} & -D_\infty^{1/2} H_{r,\infty} D_\infty^{1/2} \end{pmatrix}.$$

Note that M_∞ is symmetric. We would now like to justify the formal manipulations above and show that M_∞ is a compact operator. This will require the following two preliminary lemmas.

Lemma 1. *Let f be a conformal map from the unit disk to the region Ω with boundary Γ. Let Γ be analytic and*

$$f(e^{i\theta})/\overline{f'(e^{i\theta})} = \sum_{k=-\infty}^{\infty} h_k e^{ik\theta}.$$

Then there exists a $C > 0$ and $R < 1$ such that

$$|h_k| \le C R^{|k|}.$$

Proof: Since Γ is analytic, f extends as a bounded, analytic function with $f'(z) \neq 0$ for $|z| \le 1/R$ for some $R < 1$. Let

$$f(z) = \sum_{k=1}^{\infty} c_k z^k \quad \text{and} \quad 1/f'(z) = \sum_{j=0}^{\infty} d_j z^k.$$

Then there is a c such that $|c_k|$, $|d_k| \le c R^k$. Further, we have that

$$\begin{aligned} f(e^{i\theta})/\overline{f'(e^{i\theta})} &= \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} c_k \overline{d}_j e^{i(k-j)\theta} \\ &= \sum_{l=1}^{\infty}\sum_{j=0}^{\infty} c_{l+j}\overline{d}_j e^{il\theta} + \sum_{l=0}^{\infty}\sum_{j=l+1}^{\infty} c_{j-l}\overline{d}_j e^{-il\theta} \\ &= \sum_{l=1}^{\infty}\sum_{j=0}^{\infty} c_{l+j}\overline{d}_j e^{il\theta} + \sum_{l=0}^{\infty}\sum_{j=1}^{\infty} c_j\overline{d}_{l+j} e^{-il\theta}. \end{aligned}$$

And so

$$|h_l| = |\sum_{j=0}^{\infty} c_{l+j}\overline{d}_j| \le \sum_{j=0}^{\infty} |c_{l+j}||\overline{d}_j| \le cR^l \sum_{j=0}^{\infty} R^{2j} = \frac{cR^l}{1-R^2} = CR^l, \quad l \ge 1.$$

Similarly

$$|h_{-l}| = |\sum_{j=1}^{\infty} c_j \overline{d}_{l+j}| \le CR^l, \quad l \ge 0.$$

Next we show that the entries of $M_{r,\infty}$ and $M_{i,\infty}$ also decay exponentially fast.

Lemma 2. *Under the assumptions of Lemma 1, the (j,k)th entries of $M_{r,\infty}$ and $M_{i,\infty}$ decay like $cr^{|j+k|}$ for some $c > 0$ and $r < 1$.*

Proof: We will prove the case for $M_{r,\infty}$. The case for $M_{i,\infty}$ follows similarly. Let $m_{k,j}$ denote the (k,j)th entry of $M_{r,\infty}$. Then we must have $m_{k,j} = \sqrt{kj}\eta_{k+j-1}$. Therefore

$$|m_{k,j}| = \sqrt{kj}|h_{k+j-1}| \le C\sqrt{kj}R^{|k+j|}. \tag{11}$$

Let $r = (1+R)/2 < 1$. Since

$$\lim_{x\to\infty} \frac{1}{\sqrt{x}}(\frac{r}{R})^x = \infty,$$

there exists an $l_0 \ge 0$ such that

$$\sqrt{l}R^l \le r^l, \quad \forall l \ge l_0.$$

Let

$$c = \max_{0\le l\le l_0} \{\sqrt{l}(\frac{R}{r})^l\},$$

we then see that

$$\sqrt{l}R^l \le cr^l, \quad \forall l \ge 0.$$

The lemma now follows directly from (11).

Lemma 2 gives us the following theorem and corollary.

Theorem 1. *$M_{r,\infty} : l^1 \to l^1$ and $M_{i,\infty} : l^1 \to l^1$ are compact operators where for $\underline{y} \in l^1$,*

$$M_{r,\infty}\underline{y} = \sum_{k=1}^{\infty} \sqrt{kj}\eta_{k+j-1}y_k, \quad j = 1, 2, \ldots$$

$$M_{i,\infty}\underline{y} = \sum_{k=1}^{\infty} \sqrt{kj}\gamma_{k+j-1}y_k, \quad j = 1, 2, \ldots$$

Proof: We will prove the theorem for $M_{r,\infty}$. As above, $M_{i,\infty}$ follows similarly. Define the finite rank operators $\{M_{r,n}\} = \{D_n^{1/2}H_nD_n^{1/2}\}$ by

$$M_{r,n}\underline{y} = \sum_{k=1}^{n} \sqrt{kj}\eta_{k+j-1}y_k, \quad j = 1, 2, \ldots, n$$

for all $\underline{y} = (y_1, y_2, \ldots) \in l^1$. The goal is to show that $M_{r,\infty}$ can be approximated uniformly by these finite rank operators. (Then, e.g., a version of Theorem 4.4c [Con, p. 41] for

Banach spaces shows that $M_{r,\infty}$ is itself compact. If $A = (a_{kj})$ is an infinite matrix, then the induced l^1 operator norm is given by

$$||A||_{l^1} = \sup_j \sum_{k=1}^{\infty} |a_{kj}|;$$

see, e.g., [Con, p. 171, prob. 8]. ¿From the geometric decay of Lemma 2 we may write

$$\sum_{k=1}^{\infty} |m_{k,j}| \leq Cr^j \sum_{k=1}^{\infty} r^k \leq C_1 r^j, \quad j \geq 1.$$

Consequently,

$$\begin{aligned} ||M_{r,\infty} - M_{r,n}||_{l^1} &= \sup\left\{ \sum_{k=1}^{\infty} |m_{k,n+1}|, \sum_{k=1}^{\infty} |m_{k,n+2}|, \ldots \right\} \\ &\leq C_1 \sup\{r^{n+1}, r^{n+2}, \ldots\} \\ &= C_1 r^{n+1} \to 0. \end{aligned}$$

Thus, $M_{r,\infty}$ is compact as desired.

Corollary 1. *M_∞ is compact on $l^1 \times l^1$ where for $\underline{x} = (\underline{x}^1, \underline{x}^2) \in l^1 \times l^1$,*

$$M_\infty \begin{pmatrix} \underline{x}^1 \\ \underline{x}^2 \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{\infty} \sqrt{kj}\,\eta_{k+j-1} {x^1}_k + \sum_{k=1}^{\infty} \sqrt{kj}\,\gamma_{k+j-1} {x^2}_k \\ \sum_{k=1}^{\infty} \sqrt{kj}\,\gamma_{k+j-1} {x^1}_k - \sum_{k=1}^{\infty} \sqrt{kj}\,\eta_{k+j-1} {x^2}_k \end{pmatrix}. \qquad (12)$$

The norm on $l^1 \times l^1$ is given by

$$||\underline{x}||_{l^1 \times l^1} = ||\underline{x}^1||_{l^1} + ||\underline{x}^2||_{l^1}.$$

Proof: ¿From the notation of the problem, it is easily verified that

$$||M_\infty - M_n||_{l^1 \times l^1} \leq ||M_{r,\infty} - M_{r,n}||_{l^1} + ||M_{i,\infty} - M_{i,n}||_{l^1}$$

The result follows from Theorem 1.

Next, we discuss the discretization of (10). Since M_∞ is compact and the matrix-vector multiplications can be performed rapidly, we will solve the discrete (normal) equations using the conjugate gradient method on the subspace orthogonal to the one dimensional null space.

4. Discretization. The natural choice for discretization is to truncate the sums given in (12) to n. This will lead to finite linear systems. However, in practice one does not have the exact Fourier coefficients. If the conformal map f is known explicitly, we approximate the h_k's by evaluating $d(z) := f(z)/\overline{f'(z)}$ at the N Fourier points, $z = e^{ij\pi/N}, j = 0, 1, \cdots, N-1$, and taking the $N-$point FFT. In this case, the discrete $h_1, \cdots, h_n$, decay at a similar rate to the exact h_k (see [He, eq. 13.2-8, p. 20].) However, since the discrete Fourier coefficients are $N - periodic, h_k = h_{k-N}$, the remaining coefficients $h_{n+1} = h_{-n+1}, \cdots, h_{N-1} = h_{-1}$ do not decay geometrically. We just set $h_k = 0, k > n$ to insure geometric decay. When f is not known exactly, we use a numerical approximation at the N Fourier points given by Wegmann's method, as discussed in section 5, and again set $h_k = 0$ for $k = n+1, \cdots, N-1$. To avoid introducing more notation, we now let h_k, A_k, etc., denote the discrete Fourier coefficients.

The notation is similar to the infinite dimensional case:

$$D_n = diag(1, 2, \ldots, n),$$

$$\underline{\alpha} = (Re\ a_1, \ldots, Re\ a_n)^T, \quad \underline{\beta} = (Im\ a_1, \ldots, Im\ a_n)^T,$$

$$\underline{B} = (Re\ A_1, \ldots, Re\ A_n)^T, \quad \underline{C} = (Im\ A_1, \ldots, Im\ A_n)^T,$$

$$\underline{x} = \begin{pmatrix} D_n^{1/2}\underline{\alpha} \\ D_n^{1/2}\underline{\beta} \end{pmatrix}, \quad \underline{r} = \begin{pmatrix} D_n^{1/2}\underline{B} \\ D_n^{1/2}\underline{C} \end{pmatrix},$$

and

$$H_n = \begin{pmatrix} H_{r,n} & H_{i,n} \\ H_{i,n} & -H_{r,n} \end{pmatrix}.$$

Then analogously to the infinite system we have

$$M_n = \begin{pmatrix} M_{r,n} & M_{i,n} \\ M_{i,n} & -M_{r,n} \end{pmatrix} = \begin{pmatrix} D_n^{1/2}H_{r,n}D_n^{1/2} & D_n^{1/2}H_{i,n}D_n^{1/2} \\ D_n^{1/2}H_{i,n}D_n^{1/2} & -D_n^{1/2}H_{r,n}D_n^{1/2} \end{pmatrix},$$

so that our problem is to solve

$$(I_n + M_n)\underline{x} = \underline{r}. \tag{13}$$

Recall that $\underline{x}$ is subject to a uniqueness condition. Since $f'(0) > 0$, the condition $Im(a_1/f'(0)) = 0$ implies $x_{n+1} = 0$. Clearly, the (k, j)th entry of $M_{r,n}$ and $M_{i,n}$ are respectively $\sqrt{kj}Re(h_{k+j})$ and $\sqrt{kj}Im(h_{k+j})$.

We have computed the eigenvalues of M_n for the examples in section 5 using MATLAB. Note that if μ is an eigenvalue of M_n, then $-\mu$ is also an eigenvalue. We also find that -1 is an eigenvalue of M_n. The rest of the eigenvalues decay rapidly to 0. The decay is due to the compactness of M_∞ shown in Corollary 1 of section 3. By [An], the spectrum of M_n is near to the spectrum of M_∞ for large n. We solve the normal equations by conjugate gradient, since $(I_n + M_n)^2$ is positive semidefinite. In our examples, so far, we have noticed that $I_n + M_n$ is also positive semidefinite, but we have no proof of this, in general. In these examples, we have used conjugate gradient directly on (13) with some computational savings. (13) could also no doubt be solved efficiently with MINRES. In addtition, we have solved (a truncated version) of the nonsymmetric system (9) with GMRES with very good results. We hope to address these issues further in future work.

Recall that our infinite system (10) has a one dimensional null space. The null space is generated by the null vector,

$$\underline{v} = (-Im\ c_1, -\sqrt{2}Im\ c_2, \ldots, -\sqrt{k}Im\ c_k, \ldots, Re\ c_1, \sqrt{2}Re\ c_2, \ldots, \sqrt{k}Re\ c_k, \ldots)^T.$$

In the discrete case, we find that for large n

$$\underline{v} = (-Im\ c_1, -\sqrt{2}Im\ c_2, \ldots, -\sqrt{n}Im\ c_n, Re\ c_1, \sqrt{2}Re\ c_2, \ldots, \sqrt{n}Re\ c_n)^T$$

satisfies $(I_n + M_n)\underline{v} = \underline{0}$ to within discretization error using our discrete approximations to the c_k's. It follows that our solution can be decomposed as

$$\underline{y} = \underline{x} + \delta\underline{v}.$$

It is clear from the conjugate gradient algorithm that if the initial guess $\underline{x}^{(0)}$ is in $\underline{v}^{\perp}$ then subsequent iterates $\underline{x}^{(q)}$ will be in $\underline{v}^{\perp}$. We take $\underline{x}^{(0)} = \underline{0}$. Conjugate gradient will then find $\underline{x} \in \underline{v}^{\perp}$ and imposing the uniqueness condition $y_{n+1} = 0$ will give us δ. By the results above, $(I_n + M_n)^2$ restricted to $\underline{v}^{\perp}$ is positive definite for sufficiently large n, since the second smallest eigenvalue of $I_\infty + M_\infty$ is bounded away from 0. Therefore conjugate gradient can be applied to the normal equations and the method will converge superlinearly.

In addition, we note that the matrix-vector multiplication involving the matrix M_n can be done efficiently using FFTs. In fact, for any $n-$vector $\underline{y}$, since D_n is diagonal, $D_n^{1/2}\underline{y}$ can be computed in n operations. Moreover, the matrix-vector multiplication $H\underline{y}$, where H is the Hankel matrix $H_{r,n}$ or $H_{i,n}$, can be computed in $O(N \log N)$ by using FFTs. The idea is to compute $T\underline{s} = (HJ)(J\underline{y})$ where J is the reversion matrix with ones on the anti-diagonal and T is a Toeplitz matrix (constant along diagonals). Next we imbed T into a matrix C as follows

$$C = \begin{pmatrix} T & X \\ X & T \end{pmatrix},$$

where X is chosen to make C circulant. Now C can be decomposed as $C = F^*\Lambda F$ where F is the $N-$point Fourier matrix and Λ is a diagonal matrix containing the eigenvalues of C. For more details on fast methods for Hankel and Toeplitz matrices see, e.g., [CN].

5. Numerical examples. In examples (i), (ii), and (iii) below, we choose $\phi(\zeta) = \zeta^3$, and $\chi(\zeta) = 0$. Then $u(\eta, \mu) = \eta^4 - \mu^4$. Note that, for the conformal map $f(z)$ from the disk, $\phi(z) = (f(z))^3$ and the boundary values at the mesh points are given by $G(z) = 4(Ref(z))^3 - i4(Imf(z))^3$. The discretization error in the Tables is given by the sup norm

$$\max_{0 \le j \le N-1} |\phi(e^{i2\pi j/N}) - \phi_n(e^{i2\pi j/N})|,$$

where ϕ_n is our nth degree approximation to ϕ. For analytic curves, this error behaves similarly to the discretization error for the conformal map which is $O(R^N)$ with R as given in Lemma 1; see [De] for a discussion of the accuracy of the conformal mapping methods.

We use the FFT method in [Weg] to find the approximate conformal map f. Wegmann approximates f by solving a discrete interpolation problem on the unit disk: Find $P_{n+1}(z)$, a polynomial of degree $n+1$, such that $P_{n+1}(e^{i2\pi j/N}) \in \Gamma, j = 0, \ldots, N-1$ with the normalization that the $P_{n+1}(0)$ is fixed and the coefficients of z and z^{n+1} are real. He computes this polynomial by applying a Newton method to find a discrete approximation to the boundary correspondence. The linear systems may be solved by the conjugate gradient method in $O(N \log N)$ per step. Quadratic convergence of the Newton iterations and convergence of the polynomial to the conformal map as N increases for sufficiently smooth Γ is proven. Numerical experiments indicate that this method is among the most robust and reliable of the Fourier series methods on the disk [De].

For examples where the exact f is known, $d = f/\overline{f'}$ may be computed with either the exact or the approximate f. This seems to make little difference in the calculations if the approximate f is sufficiently accurate. The timings for finding the approximate f using [Weg] are usually only slightly greater than the timings given in the Tables for solving the boundary value problem for a given N.

In the Tables below, *iter* is the number of iterations required by conjugate gradient for the residuals to be $\leq 10^{-14}$. The computations were done in double precision on the WSU IBM ES9121 Model 440 mainframe computer and some rough timings are given. (Figures 1 and 2 and some of our examples were also done in MATLAB with similar results.) Stopping the iterations after the level of discretization error has been achieved could further reduce the timings, though not dramatically for these examples of very fast superlinear convergence. Note that as the minor-to-major axis ratio α of a region decreases toward 0 (that is, as R in Lemma 1 increases to 1), the convergence rate of the conjugate gradient method decreases. In our examples below, R may be taken as the distance from the origin to the nearest singularity of f and the connection with the minor-to-major aspect ratio is known [De]. In a future paper we will show how the convergence rate of the conjugate gradient method depends on the smoothness of the boundary Γ.

Other cases were also tried successfully, such as the simple examples in [KK]. If the biharmonic function has too simple a Goursat representation, the iterations may converge artificially fast. For instance, if $u(\eta, \mu) = \eta^2 + \eta\mu + \mu^2$, then $\phi(z) = f(z)$ and convergence is achieved in one iteration if N is large enough. On the other hand, note that the 5-to-1 ellipse in [GGMa, Table 3] is a difficult region for the conformal map from the disk and would require large N.

Example (i), inverted ellipse. Here $\Gamma : \gamma(\sigma) = \rho(\sigma)e^{i\sigma}$ where $\rho(\sigma) = \sqrt{1 - (1-\alpha^2)\sin^2\sigma}$ for $0 \leq \sigma \leq 2\pi$ and $0 < \alpha \leq 1$. This map is derived by inverting the familiar Joukowski map to the exterior of an ellipse. We have

$$f(z) = \frac{2\alpha z}{1 + \alpha - (1-\alpha)z^2}.$$

See Table 1 for results. Notice how *iter* is roughly independent of N but increases with α in our examples.

Table 1. Inverted ellipse with exact map and conjugate gradient

α	N	discr. error	iter	CPU sec
.8	32	$.5 \cdot 10^{-5}$	4	.2
.8	64	$.5 \cdot 10^{-12}$	3	.2
.8	128	$.5 \cdot 10^{-14}$	3	.2
.4	64	$.1 \cdot 10^{-3}$	6	.2
.4	128	$.6 \cdot 10^{-9}$	4	.2
.4	256	$.1 \cdot 10^{-13}$	4	.3
.2	128	$.2 \cdot 10^{-3}$	6	.3
.2	256	$.2 \cdot 10^{-8}$	4	.3
.2	512	$.3 \cdot 10^{-13}$	4	.4

Example (ii), arctanh. Here the conformal map is given by $f(z) = \log((1+rz)/(1-rz))$, $0 < r < 1$, which maps the disk to increasingly elongated, cigar-shaped regions as $r \uparrow 1$. This map is perhaps the simplest example of a conformal map exhibiting the exponential crowding [De]. Figure 1 shows roughly 7 outlying eigenvalues of $(I_n + M_n)^2$ for $\alpha = .49$. Thus conjugate gradient for the normal equations takes about 7 iterations to converge as one would expect; see Table 2 and Figure 2. Also note the semilog plot in Figure 2 that shows the superlinear convergence behavior of the residuals (using MATLAB).

Table 2. Arctanh regions with exact map and conjugate gradient

$\alpha(r)$	N	discr. error	iter	CPU sec
.84 (.5)	32	$.4 \cdot 10^{-4}$	6	.2
.84 (.5)	64	$.4 \cdot 10^{-9}$	4	.2
.84 (.5)	128	$.8 \cdot 10^{-14}$	4	.2
.49 (.9)	128	$.6 \cdot 10^{-3}$	8	.4
.49 (.9)	256	$.4 \cdot 10^{-6}$	6	.3
.49 (.9)	512	$.4 \cdot 10^{-12}$	6	.5
.29 (.99)	512	$.4 \cdot 10^{-1}$	14	.9
.29 (.99)	1024	$.2 \cdot 10^{-2}$	13	1.5
.29 (.99)	2048	$.6 \cdot 10^{-5}$	12	2.8
.29 (.99)	4096	$.1 \cdot 10^{-9}$	12	6.0

Example (iii), ellipse. Here $\Gamma : \gamma(\sigma) = \rho(\sigma)e^{i\sigma}$ where $\rho(\sigma) = \alpha/\sqrt{1-(1-\alpha^2)\cos^2\sigma}$ for $0 \leq \sigma \leq 2\pi$ and $0 < \alpha \leq 1$. The exact map can be given in terms of an elliptic integral.

This case also exhibits exponential crowding [De]. We approximate the f with [Weg].

Table 3. Ellipses with approximate map and conjugate gradient

α	N	discr. error	iter	CPU sec
.8	32	$.7 \cdot 10^{-3}$	6	.2
.8	64	$.5 \cdot 10^{-6}$	4	.2
.8	128	$.4 \cdot 10^{-12}$	3	.2
.8	256	$.2 \cdot 10^{-13}$	3	.3
.6	128	$.6 \cdot 10^{-3}$	8	.3
.6	256	$.3 \cdot 10^{-6}$	6	.4
.6	512	$.2 \cdot 10^{-12}$	3	.4
.4	2048	$.2 \cdot 10^{-4}$	10	2.6
.4	4096	$.7 \cdot 10^{-10}$	8	4.6

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Figure captions

Figure 1. Eigenvalue distribution for $(I_n + M_n)^2$ for the arctanh region, example (ii), with $\alpha = .49$ and $N = 256$.

Figure 2. Convergence of residuals for 7 iterations of the conjugate gradient method for the normal equations for the arctanh region, example (ii), with $\alpha = .49$ and $N = 256$.