Circulant Preconditioners for Ill-Conditioned Boundary Integral Equations from Potential Equations*

Raymond H. Chan† Hai-Wei Sun‡ Wing-Fai Ng§

Abstract

In this paper, we consider solving potential equations by the boundary integral equation approach. The equations so derived are Fredholm integral equations of the first kind and are known to be ill-conditioned. Their discretized matrices are dense and have condition numbers growing like $O(n)$ where n is the matrix size. We propose to solve the equations by the preconditioned conjugate gradient method with circulant integral operators as preconditioners. These are convolution operators with periodic kernels and hence can be inverted efficiently by using fast Fourier transforms. We prove that the preconditioned systems are well-conditioned, and hence the convergence rate of the method is linear. Numerical results for two types of regions are given to illustrate the fast convergence.

Key Words. Boundary integral equations, Fredholm integral equations, circulant preconditioners, preconditioned conjugate gradient method.

AMS(MOS) subject classifications. 45B05, 65F10, 65R20.

1 Introduction

In this paper, we study the solution of the potential equation

$$\begin{cases} \Delta w(x) = 0, & x \in \Omega, \\ w(x) = g(x), & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\partial\Omega$ is a smooth closed curve in $\mathbb{R}^2$ and Ω is either the bounded interior region with boundary $\partial\Omega$ or the unbounded exterior region with boundary $\partial\Omega$. In the boundary integral equation approach, see for instance Chen and Zhou [9, §6.12], the harmonic function $w(x)$ is represented as a single-layer potential generated by a source distribution $\sigma(x)$ over $\partial\Omega$, with the potential satisfying the boundary condition $g(x)$ prescribed for $w(x)$. More precisely, we write

$$w(x) = -\frac{1}{2\pi}\int_{\partial\Omega} \log|x-y|\sigma(y)dS_y + \eta, \qquad x \in \Omega \tag{2}$$

where S_y is the arc length variable corresponding to y and η is a constant to be determined.

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‡Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong.

§Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong.

The limit of (2), as x is taken to $\partial\Omega$, gives the integral equation on the boundary:

$$g(x) = -\frac{1}{2\pi}\int_{\partial\Omega} \log|x-y|\sigma(y)dS_y + \eta, \quad x \in \partial\Omega. \tag{3}$$

We will see in §2 that $\sigma(y)$ and η can be obtained by solving the boundary integral equations

$$-\frac{1}{2\pi}\int_{\partial\Omega} \log|x-y|\sigma_i(y)dS_y = g_i(x), \qquad x \in \partial\Omega, \qquad i = 1, 2, \tag{4}$$

for $\sigma_1(y)$ and $\sigma_2(y)$ with $g_1(x) = g(x)$ and $g_2(x) \equiv 1$. Once $\sigma(y)$ and η are obtained, values of $w(x)$ in Ω can be computed from (2).

The well-known advantage of the boundary integral equation approach is that the dimension of the problem is reduced by one. The discrete matrix A_n associated with the integral equation (4) is only of size n-by-n for mesh size proportional to $1/n$. In contrast, the discrete matrix for the partial differential equation (1) will be of size n^2-by-n^2. However, the major drawback of the approach is that A_n is a dense matrix. Hence solving the problem with a direct method will require $O(n^3)$ operations which will be too expensive for practical, large scale numerical computations.

If an iterative method, such as the conjugate gradient method (see Golub and van Loan [11, p.524]) is used, then the cost per iteration is dominated by the cost of multiplying A_n to a vector say $\mathbf{x}$. This in general requires $O(n^2)$ operations. For some special boundaries $\partial\Omega$, the product $A_n\mathbf{x}$ may be obtained fast. As an example, if $\partial\Omega$ is an ellipse, A_n will be a circulant-plus-Hankel matrix and $A_n\mathbf{x}$ can be computed in $O(n \log n)$ operations, see Example 1 in §5. In general, we can use fast multiplication schemes [1, 6, 12] to reduce the cost of multiplication to $O(n \log n)$ operations depending on the smoothness of the boundary.

Another difficulty for the boundary integral equation (4) is that it is a Fredholm equation of the first kind with a weakly singular kernel, see Kress [15, p.270]. The matrix A_n therefore is known to be ill-conditioned. In §2, we will see that the condition number of A_n increases like $O(n)$. Therefore if the system is solved by the conjugate gradient method, the number of iterations required for convergence will be increasing like $O(n^{1/2})$, see Golub and van Loan [11, Theorem 10.2.5] and also the numerical results in §5.

One standard way to speed up the convergence rate in the conjugate gradient method is to apply a preconditioner and then solve the preconditioned system, see Golub and van Loan [11, p.529]. Gohberg, Hanke and Koltracht in [10] have considered using circulant integral operators to precondition Wiener-Hopf integral equations of the second kind defined on $[0, \infty)$. Circulant integral operators are convolution operators with periodic kernels. Their inverses (both for the continuous operators and for the discretized matrices) can be obtained efficiently using Fourier transforms. For Wiener-Hopf equations of the second kind on $[0, \infty)$, which are convolution-type, well-conditioned integral equations, Gohberg *et. al.* showed in [10] that with preconditioning, the convergence rate can be increased from linear to superlinear.

In this paper, we will consider the use of circulant integral operators as preconditioners for integral equations of the first kind as given in (4). Our integral equations are not of convolution-type and are ill-conditioned. We will show that the preconditioned systems will become well-conditioned and therefore the convergence rate is linear. In particular, the number of iterations required for convergence is reduced from $O(n^{1/2})$ to $O(1)$ if our proposed preconditioner is used.

We remark that the discretization matrices of circulant integral operators using the rectangular quadrature rule are circulant matrices, see for instance Chan, Jin and Ng [4]. Circulant matrices have been proposed and used as preconditioners for Toeplitz matrices in the past ten years, see the survey paper by Chan and Ng [7] and the references therein. It is established theoretically that the circulant-preconditioned systems converge superlinearly when the given Toeplitz system is well-conditioned, see for instance Chan and Strang [8]. However, the performance of circulant preconditioners for ill-conditioned Toeplitz systems is not good in general and in fact circulant preconditioners fail in some cases, see Chan [3]. For these systems, band-Toeplitz type preconditioners have been proven to make the ill-conditioned problems well-conditioned, see for instance [3]. We emphasize that in this paper, the discrete matrices A_n we considered are ill-conditioned and are *not* Toeplitz matrices. But our results imply that they can be preconditioned by circulant matrices to obtain well-conditioned systems.

One kind of circulant preconditioner that has been investigated extensively in the study of preconditioners for Toeplitz matrices is the *optimal circulant preconditioner* proposed by T. Chan in [2]. It can be defined for arbitrary matrices. T. Chan's idea of constructing optimal circulant preconditioners has been incorporated in Gohberg, Hanke and Koltracht [10] in developing optimal circulant integral operators for convolution-type integral operators. Chan and Lin [5] later extended the idea to develop optimal circulant integral operators for general non-convolution type integral operators. In this paper, we will concentrate on the use of optimal circulant integral operators for (4).

The outline of the paper is as follows. In §2, we show the equivalence of the equations (3) and (4) and give some properties of (4). In particular, we note that the discretized systems of (4) will be ill-conditioned. In §3, we introduce the optimal circulant integral operator for (4) and study some properties of its associated bilinear form. In §4 we show that the condition numbers of the discretized circulant-preconditioned systems are uniformly bounded independent of the size of the systems. Numerical results for two types of domains are given in §5 to illustrate the fast convergence of our method and some concluding remarks are given in §6.

2 The Boundary Integral Operator

In this section, we review some basic results of boundary integral equations and of the boundary integral operators they induce. In particular, we show that the density function $\sigma(y)$ in (3) can be obtained by solving $\sigma_1(y)$ and $\sigma_2(y)$ in (4) and that the boundary integral operator induced by (4) is ill-conditioned. These results are well-known but we recall them here for completeness.

We first note that, besides (3), the density function $\sigma(y)$ for the potential equation (1) must also satisfy the consistency condition

$$\int_{\partial\Omega} \sigma(y) dS_y = \chi,$$

where χ determines the growth of the harmonic function $w(x)$ at infinity, see Chen and Zhou [9, Theorems 6.12.1 and 6.12.5–6]. For exterior problems where the growth at infinity is $O(1)$ or for interior problems, we have $\chi = 0$. In the case where the growth of $w(x)$ at infinity is of order $\chi \log |x| + O(1)$, we can define the new variable $\tilde{w}(x) = w(x) - \chi \log |x|$ to eliminate χ. Thus without loss of generality, we assume in the following that the consistency condition is

$$\int_{\partial\Omega} \sigma(y) dS_y = 0. \tag{5}$$

Another thing we can assume without loss of generality is that

$$\mathrm{diam}(\partial\Omega) \equiv \max_{x,y\in\partial\Omega} |x-y| < 1. \tag{6}$$

For if (6) does not hold, we can always make a scaling transformation to reduce the diameter. More precisely, let $\partial\Omega_\rho = \{\rho \cdot x \mid x \in \partial\Omega\}$ with $\rho > 0$. Obviously, $\partial\Omega_\rho$ will satisfy (6) if ρ is properly chosen. We remark that (6) is used to guarantee that (4) is uniquely solvable, see Theorem 1 below.

We note that in the transformed domain Ω_ρ, (3) and (5) still holds. In fact, using (5), we have

$$\begin{aligned} -\frac{1}{2\pi}\int_{\partial\Omega} \log|x-y|\sigma(y)dS_y &= -\frac{1}{2\pi}\int_{\partial\Omega} \log|\rho x-\rho y|\sigma(y)dS_y + \frac{1}{2\pi}\int_{\partial\Omega} \log\rho\cdot\sigma(y)dS_y \\ &= -\frac{1}{2\pi}\int_{\partial\Omega_\rho} \log|\tilde{x}-\tilde{y}|\tilde{\sigma}(\tilde{y})dS_{\tilde{y}} + \frac{1}{2\pi}\log\rho\int_{\partial\Omega}\sigma(y)dS_y \\ &= -\frac{1}{2\pi}\int_{\partial\Omega_\rho} \log|\tilde{x}-\tilde{y}|\tilde{\sigma}(\tilde{y})dS_{\tilde{y}}, \end{aligned}$$

where $\tilde{x} = \rho x$, $\tilde{y} = \rho y$ and $\tilde{\sigma}(\tilde{y}) = \sigma(y)/\rho$. Thus (3) is equivalent to

$$g\left(\frac{\tilde{x}}{\rho}\right) = -\frac{1}{2\pi}\int_{\partial\Omega_\rho} \log|\tilde{x}-\tilde{y}|\tilde{\sigma}(\tilde{y})dS_{\tilde{y}} + \eta.$$

Clearly (5) is equivalent to $\int_{\partial\Omega_\rho}\tilde{\sigma}(\tilde{y})dS_{\tilde{y}} = 0$. Thus in the following, we assume without loss of generality that (3) and (5) hold in a domain that satisfies (6).

To find $\sigma(y)$ in (3), we first solve (4) for $\sigma_1(y)$ and $\sigma_2(y)$. Then it is straightforward to verify that $\sigma(y)$ is given by

$$\sigma(y) = \sigma_1(y) - \eta\sigma_2(y), \tag{7}$$

where by (5), η is given by

$$\eta = \frac{\int_{\partial\Omega}\sigma_1(y)dS_y}{\int_{\partial\Omega}\sigma_2(y)dS_y}. \tag{8}$$

We note that the denominator $\int_{\partial\Omega}\sigma_2(y)dS_y$ cannot be zero, see Chen and Zhou [9, p.287]. Once $\sigma(y)$ and η are obtained, values of $w(x)$ in Ω can be computed from (2). We remark that $\sigma_1(y)$ and $\sigma_2(y)$ in (4) are not required to satisfy the consistency condition (5). The consistency condition on $\sigma(y)$ is satisfied by the proper choice of η in (8).

Corresponding to (4), we define the boundary integral operator

$$(\mathcal{A}u)(x) \equiv -\frac{1}{2\pi}\int_{\partial\Omega}\log|x-y|u(y)dS_y, \qquad x\in\partial\Omega. \tag{9}$$

We will use $\langle\cdot,\cdot\rangle$ to denote the inner product on $\mathbf{L}^2(\partial\Omega)\times\mathbf{L}^2(\partial\Omega)$. We now recall the well-known result that $\mathcal{A}$ defines a continuous positive definite symmetric bilinear form on $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ provided that (6) holds. We recall that (6) holds in general by scaling the region if necessary.

Theorem 1 (Hsiao and Wendland [13], Chen and Zhou [9, Remark 6.8.1]) *Suppose (6) holds. Then the bilinear form*

$$\langle\mathcal{A}u,v\rangle = -\frac{1}{2\pi}\int_{\partial\Omega}\int_{\partial\Omega}\log|x-y|u(y)\overline{v(x)}dS_ydS_x \qquad \forall u,v\in\mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \tag{10}$$

is continuous, positive definite, symmetric on $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$*. More precisely, there are constants* $\alpha \geq \beta > 0$*, such that*

$$\langle \mathcal{A}v, v\rangle \geq \alpha \|v\|^2_{\mathbf{H}^{-\frac{1}{2}}}, \qquad \forall v \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \tag{11}$$

and

$$\langle \mathcal{A}u, v\rangle \leq \beta \|u\|_{\mathbf{H}^{-\frac{1}{2}}} \|v\|_{\mathbf{H}^{-\frac{1}{2}}}, \qquad \forall u, v \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega). \tag{12}$$

In view of the theorem, we have for all $v \in \mathbf{H}^0(\partial\Omega) = \mathbf{L}^2(\partial\Omega)$,

$$\alpha \frac{\|v\|^2_{\mathbf{H}^{-\frac{1}{2}}}}{\|v\|^2_{\mathbf{H}^0}} \leq \frac{\langle \mathcal{A}v, v\rangle}{\langle v, v\rangle} \leq \beta \frac{\|v\|^2_{\mathbf{H}^{-\frac{1}{2}}}}{\|v\|^2_{\mathbf{H}^0}}.$$

Since the identity mapping from $\mathbf{H}^0(\partial\Omega)$ to $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ involves a loss of a half derivative, we expect the ratio $\|v\|^2_{\mathbf{H}^0}/\|v\|^2_{\mathbf{H}^{-\frac{1}{2}}}$ and hence the condition number of the discretized matrix of $\mathcal{A}$ to be growing like $O(n)$ where n is the size of the discretization matrix. Thus if the conjugate gradient method is employed to solve the system, we expect the number of iterations required for convergence grows like $O(\sqrt{n})$, see Golub and van Loan [11, Theorem 10.2.5] and also the numerical results in §5.

For simplicity, we parametrize the boundary $\partial\Omega$ as $(x_1(\theta), x_2(\theta))$, $0 \leq \theta \leq 2\pi$. Then the bilinear form in (10) can be rewritten as

$$\langle \mathcal{A}u, v\rangle = \int_0^{2\pi}\int_0^{2\pi} a(\theta, \phi) u(\theta) \overline{v(\phi)} d\theta d\phi \tag{13}$$

with kernel function

$$a(\theta, \phi) = -\frac{1}{4\pi} \log\left\{(x_1(\theta) - x_1(\phi))^2 + (x_2(\theta) - x_2(\phi))^2\right\}. \tag{14}$$

As $\partial\Omega$ is a closed curve, $a(\theta, \phi)$ is clearly 2π-periodic in both arguments. Since $\partial\Omega$ is smooth, the functions $x_1(\cdot)$ and $x_2(\cdot)$ are smooth. We can write

$$a(\theta, \phi) = -\frac{1}{2\pi} \log|(\theta - \phi)(2\pi - |\theta - \phi|)| - \frac{1}{4\pi} \log\left\{\frac{(x_1(\theta) - x_1(\phi))^2 + (x_2(\theta) - x_2(\phi))^2}{(\theta - \phi)^2 (2\pi - |\theta - \phi|)^2}\right\}. \tag{15}$$

Using the smoothness of $x_1(\cdot)$ and $x_2(\cdot)$, we have for all $\phi \in [0, 2\pi]$ and $j = -1, 0, 1$,

$$\left|\lim_{\theta \to \phi + 2j\pi} \log\left\{\frac{(x_1(\theta) - x_1(\phi))^2 + (x_2(\theta) - x_2(\phi))^2}{(\theta - \phi)^2 (2\pi - |\theta - \phi|)^2}\right\}\right| = \left|\log\left\{\frac{(x_1'(\phi))^2 + (x_2'(\phi))^2}{4\pi^2}\right\}\right| < \infty.$$

Hence the second term in the right hand side of (15) is continuous in $[0, 2\pi]^2$. In particular, it is a function in $\mathbf{L}^2[0, 2\pi]^2$. Clearly the first term in the right hand side of (15) is also in $\mathbf{L}^2[0, 2\pi]^2$. Thus $a(\theta, \phi)$ is in $\mathbf{L}^2[0, 2\pi]^2$.

3 The Optimal Circulant Integral Operator

One way of overcoming the ill-conditioned nature of the operator $\mathcal{A}$ is to use a preconditioner. Here we consider the use of circulant integral operators. These are integral operators of the form

$$(\mathcal{B}u)(\phi) = \int_0^{2\pi} b(\theta - \phi) u(\theta) d\theta, \qquad 0 \leq \phi \leq 2\pi$$

with 2π-periodic kernel function $b(\phi)$. The *optimal circulant integral operator* for $\mathcal{A}$ is the unique circulant integral operator $\mathcal{C}$ that minimizes the Hilbert-Schmidt norm $|||\mathcal{B}-\mathcal{A}|||$ over all circulant integral operators $\mathcal{B}$, where

$$|||\mathcal{B}-\mathcal{A}|||^2 \equiv \int_0^{2\pi}\int_0^{2\pi} |a(\theta,\phi)-b(\theta-\phi)|^2 d\theta d\phi, \tag{16}$$

see Gohberg *et. al.* [10].

Recall that for our boundary integral operator $\mathcal{A}$ as defined in (9), its kernel function $a(\cdot,\cdot)$ is 2π-periodic in both arguments and is in $\mathbf{L}^2[0,2\pi]^2$ (cf. (15)). Therefore

$$c(\phi) \equiv \frac{1}{2\pi}\int_0^{2\pi} a(\theta,\theta-\phi)d\theta, \qquad 0<\phi<2\pi, \tag{17}$$

is a 2π-periodic function and is in $\mathbf{L}^2[0,2\pi]$. Hence

$$(\mathcal{C}u)(\phi) \equiv \int_0^{2\pi} c(\theta-\phi)u(\theta)d\theta, \qquad 0\le\phi\le 2\pi \tag{18}$$

is a circulant integral operator. Moreover, we note that $\{e^{ij\theta}\}_{j\in\mathbb{Z}}$ forms a complete set of eigenfunctions of $\mathcal{C}$.

We now show that $\mathcal{C}$ as defined in (18) is the optimal circulant integral operator of $\mathcal{A}$. For this, we need the following lemma which will also be useful in the next section in analyzing the convergence rate of the preconditioned systems.

Lemma 1 *Let $\mathcal{A}$ be given as in (9) and $\mathcal{C}$ be the circulant integral operator as defined in (17) and (18). Then*

$$\langle \mathcal{A}e^{ij\theta}, e^{ij\phi}\rangle = \langle \mathcal{C}e^{ij\theta}, e^{ij\phi}\rangle, \qquad j=0,\pm1,\pm2,\cdots. \tag{19}$$

Proof: By (13), we have, for all integers j,

$$\langle \mathcal{A}e^{ij\theta}, e^{ij\phi}\rangle = \int_0^{2\pi}\int_0^{2\pi} a(\theta,\phi)e^{ij(\theta-\phi)}d\theta d\phi = \int_0^{2\pi}\int_{\theta-2\pi}^{\theta} a(\theta,\theta-\phi)e^{ij\phi}d\phi d\theta.$$

Since $a(\cdot,\cdot)$ is 2π-periodic in both arguments, we have

$$\langle \mathcal{A}e^{ij\theta}, e^{ij\phi}\rangle = \int_0^{2\pi}(\int_0^{2\pi} a(\theta,\theta-\phi)d\theta)e^{ij\phi}d\phi = 2\pi\int_0^{2\pi} c(\phi)e^{ij\phi}d\phi, \tag{20}$$

where the last equality follows from (17). On the other hand, by (18), we have

$$\langle \mathcal{C}e^{ij\theta}, e^{ij\phi}\rangle = \int_0^{2\pi}\int_0^{2\pi} c(\theta-\phi)e^{ij(\theta-\phi)}d\theta d\phi = \int_0^{2\pi}\int_{\theta-2\pi}^{\theta} c(\phi)e^{ij\phi}d\phi d\theta.$$

Since $c(\phi)$ is 2π-periodic, we then have

$$\langle \mathcal{C}e^{ij\theta}, e^{ij\phi}\rangle = \int_0^{2\pi}\int_0^{2\pi} c(\phi)e^{ij\phi}d\phi d\theta = 2\pi\int_0^{2\pi} c(\phi)e^{ij\phi}d\phi.$$

Comparing this with (20), we have (19). $\square$

Now we are ready to prove that $\mathcal{C}$ in (18) is the optimal circulant integral operator for $\mathcal{A}$ defined in (9).

Lemma 2 *The operator $\mathcal{C}$ in (18) with kernel function (17) is the optimal circulant integral operator for $\mathcal{A}$ defined in (9), i.e. it minimizes the Hilbert-Schmidt norm $|||\mathcal{B}-\mathcal{A}|||$ over all circulant integral operators $\mathcal{B}$.*

Proof: Since $a(\theta,\phi) \in \mathbf{L}^2[0,2\pi]^2$, we can write, by using Fourier expansions,

$$a(\theta,\phi) = \sum_{j,l=-\infty}^{\infty} \langle \mathcal{A}e^{ij\theta}, e^{il\phi}\rangle \; e^{ij\theta}e^{-il\phi}, \qquad 0 \le \theta,\phi \le 2\pi, \tag{21}$$

For any circulant integral operator $\mathcal{B}$, since its kernel function $b(\theta-\phi)$ is 2π-periodic, we have

$$b(\theta-\phi) = \sum_{j=-\infty}^{\infty} \langle \mathcal{B}e^{ij\theta}, e^{ij\phi}\rangle \; e^{ij\theta}e^{-ij\phi}, \qquad 0 \le \theta,\phi \le 2\pi.$$

Combining this with (21) and using the orthogonality of $\{e^{ij\theta}\}_{j\in\mathbb{Z}}$, we can rephrase the Hilbert-Schmidt norm in (16) as

$$|||\mathcal{A}-\mathcal{B}|||^2 = \sum_{j=-\infty}^{\infty} |\langle \mathcal{A}e^{ij\theta}, e^{ij\phi}\rangle - \langle \mathcal{B}e^{ij\theta}, e^{ij\phi}\rangle|^2 + \sum_{\substack{j,l=-\infty \\ j\neq l}}^{\infty} |\langle \mathcal{A}e^{ij\theta}, e^{il\phi}\rangle|^2.$$

Clearly, the expression becomes minimal if and only if

$$\langle \mathcal{B}e^{ij\theta}, e^{ij\phi}\rangle = \langle \mathcal{A}e^{ij\theta}, e^{ij\phi}\rangle, \quad \forall j \in \mathbb{Z}.$$

Thus the result follows from Lemma 1. $\square$

Before discussing the next lemma, which will be useful in the next section, it is worth noting that for any function $v \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$, there exists a sequence of scalars $\{v_j\}_{j\in\mathbb{Z}}$ such that

$$v(\theta) = \sum_{j=-\infty}^{\infty} v_j e^{ij\theta}$$

and

$$\|v\|^2_{\mathbf{H}^{-\frac{1}{2}}} = \sum_{j=-\infty}^{\infty} (1+j^2)^{-\frac{1}{2}} |v_j|^2 < \infty,$$

see Kress [15, Theorem 8.9]. This characterization of $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ is useful in the following lemma.

Lemma 3 *Let $u, v \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ with expansions*

$$u(\theta) = \sum_{j=-\infty}^{\infty} u_j e^{ij\theta} \qquad \text{and} \qquad v(\phi) = \sum_{l=-\infty}^{\infty} v_l e^{il\phi},$$

that satisfy

$$\|u\|^2_{\mathbf{H}^{-\frac{1}{2}}} = \sum_{j=-\infty}^{\infty} (1+j^2)^{-\frac{1}{2}} |u_j|^2 < \infty \qquad \text{and} \qquad \|v\|^2_{\mathbf{H}^{-\frac{1}{2}}} = \sum_{l=-\infty}^{\infty} (1+l^2)^{-\frac{1}{2}} |v_l|^2 < \infty.$$

Then

$$\langle \mathcal{C}u, v\rangle = \sum_{j,l=-\infty}^{\infty} u_j \overline{v_l} \langle \mathcal{C}e^{ij\theta}, e^{il\phi}\rangle = \sum_{j=-\infty}^{\infty} u_j \overline{v_j} \langle \mathcal{C}e^{ij\theta}, e^{ij\phi}\rangle. \tag{22}$$

Proof: Recall from (17) that the kernel function $c(\phi)$ of $\mathcal{C}$ is a 2π-periodic function in $\mathbf{L}^2[0,2\pi]$. Therefore $\{e^{ij\theta}\}_{j\in\mathbb{Z}}$ forms a complete orthonormal set of eigenfunctions of $\mathcal{C}$. From this, the last equality of (22) follows. By the dominated convergence theorem (cf. Rudin [16, p.26]), the first equality in (22) holds if we can show that

$$\sum_{j,l=-\infty}^{\infty} |u_j \overline{v_l} \langle \mathcal{C} e^{ij\theta}, e^{il\phi}\rangle| < \infty.$$

However, by (19), we have

$$\sum_{j,l=-\infty}^{\infty} |u_j \overline{v_l} \langle \mathcal{C} e^{ij\theta}, e^{il\phi}\rangle| = \sum_{j=-\infty}^{\infty} |u_j v_j \langle \mathcal{C} e^{ij\theta}, e^{ij\phi}\rangle| = \sum_{j=-\infty}^{\infty} |u_j v_j \langle \mathcal{A} e^{ij\theta}, e^{ij\phi}\rangle|.$$

Hence by (12), we have

$$\sum_{j,l=-\infty}^{\infty} |u_j \overline{v_l} \langle \mathcal{C} e^{ij\theta}, e^{il\phi}\rangle| \le \beta \sum_{j=-\infty}^{\infty} |u_j||v_j| \|e^{ij\theta}\|^2_{\mathbf{H}^{-\frac{1}{2}}} = \beta \sum_{j=-\infty}^{\infty} (1+j^2)^{-\frac{1}{2}} |u_j||v_j|,$$

as $\|e^{ij\theta}\|^2_{\mathbf{H}^{-\frac{1}{2}}} = (1+j^2)^{-\frac{1}{2}}$. By using the Cauchy-Schwartz inequality, we then get

$$\begin{aligned} \sum_{j,l=-\infty}^{\infty} |u_j \overline{v_l} \langle \mathcal{C} e^{ij\theta}, e^{il\phi}\rangle| &\le \beta \left\{ \sum_{j=-\infty}^{\infty} (1+j^2)^{-\frac{1}{2}} |u_j|^2 \right\}^{1/2} \left\{ \sum_{l=-\infty}^{\infty} (1+l^2)^{-\frac{1}{2}} |v_l|^2 \right\}^{1/2} \\ &= \beta \|u\|_{\mathbf{H}^{-\frac{1}{2}}} \|v\|_{\mathbf{H}^{-\frac{1}{2}}} < \infty. \qquad \square \end{aligned}$$

4 Condition Numbers of the Preconditioned Systems

In this section, we study the spectrum of the preconditioned operator $\mathcal{C}^{-1}\mathcal{A}$ and show that the Galerkin approximation of the preconditioned operator results in well-conditioned discrete systems. We begin by showing that the optimal circulant integral operator is also positive definite and continuous.

Theorem 2 *The optimal circulant integral operator $\mathcal{C}$ of $\mathcal{A}$ satisfies*

$$\langle \mathcal{C} v, v\rangle \ge \alpha \|v\|^2_{\mathbf{H}^{-\frac{1}{2}}}, \qquad \forall v \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \tag{23}$$

and

$$\langle \mathcal{C} u, v\rangle \le \beta \|u\|_{\mathbf{H}^{-\frac{1}{2}}} \|v\|_{\mathbf{H}^{-\frac{1}{2}}}, \qquad \forall u, v \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \tag{24}$$

where α and β are given by (11) and (12).

Proof: Let $v \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ with expansion

$$v(\theta) = \sum_{j=-\infty}^{\infty} v_j e^{ij\theta}$$

which satisfies

$$\|v\|^2_{\mathbf{H}^{-\frac{1}{2}}} = \sum_{j=-\infty}^{\infty} (1+j^2)^{-\frac{1}{2}} |v_j|^2 < \infty.$$

By (22), (19) and (11), we have

$$\langle \mathcal{C}v, v\rangle = \sum_{j=-\infty}^{\infty} |v_j|^2 \langle \mathcal{C}e^{ij\theta}, e^{ij\phi}\rangle = \sum_{j=-\infty}^{\infty} |v_j|^2 \langle \mathcal{A}e^{ij\theta}, e^{ij\phi}\rangle \geq \alpha \sum_{j=-\infty}^{\infty} |v_j|^2 \|e^{ij\theta}\|^2_{\mathbf{H}^{-\frac{1}{2}}}.$$

Since $\|e^{ij\theta}\|^2_{\mathbf{H}^{-\frac{1}{2}}} = (1+j^2)^{-\frac{1}{2}}$, we then have

$$\langle \mathcal{C}v, v\rangle \geq \alpha \sum_{j=-\infty}^{\infty} (1+j^2)^{-\frac{1}{2}} |v_j|^2 = \alpha \|v\|^2_{\mathbf{H}^{-\frac{1}{2}}}.$$

Similarly, for any $u, v \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$, by (22) and (19) again, we also have

$$\langle \mathcal{C}u, v\rangle = \sum_{j=-\infty}^{\infty} u_j \overline{v_j} \langle \mathcal{C}e^{ij\theta}, e^{ij\phi}\rangle = \sum_{j=-\infty}^{\infty} u_j \overline{v_j} \langle \mathcal{A}e^{ij\theta}, e^{ij\phi}\rangle.$$

Hence by (12), we have

$$\langle \mathcal{C}u, v\rangle \leq \beta \sum_{j=-\infty}^{\infty} |u_j||v_j| \|e^{ij\theta}\|^2_{\mathbf{H}^{-\frac{1}{2}}} = \beta \sum_{j=-\infty}^{\infty} (1+j^2)^{-\frac{1}{2}} |u_j||v_j| \leq \beta \|u\|_{\mathbf{H}^{-\frac{1}{2}}} \|v\|_{\mathbf{H}^{-\frac{1}{2}}},$$

where the last equality follows by using the Cauchy-Schwartz inequality. □

As a consequence, we immediately have a bound on the spectrum of $\mathcal{C}^{-1}\mathcal{A}$.

Theorem 3 *There exist positive constants $\gamma_2 \geq \gamma_1 > 0$ such that the spectrum of $\mathcal{C}^{-1}\mathcal{A}$ lies in $[\gamma_1, \gamma_2]$.*

Proof: For any $v \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$, we have, in view of (11), (12), (23) and (24),

$$\alpha \|v\|^2_{\mathbf{H}^{-\frac{1}{2}}} \leq \langle \mathcal{A}v, v\rangle \leq \beta \|v\|^2_{\mathbf{H}^{-\frac{1}{2}}}$$

and

$$\alpha \|v\|^2_{\mathbf{H}^{-\frac{1}{2}}} \leq \langle \mathcal{C}v, v\rangle \leq \beta \|v\|^2_{\mathbf{H}^{-\frac{1}{2}}}.$$

Hence

$$\frac{\alpha}{\beta} \leq \frac{\langle \mathcal{A}v, v\rangle}{\langle \mathcal{C}v, v\rangle} \leq \frac{\beta}{\alpha}. \qquad \square$$

Next we consider the condition number of the discretized systems. Let $\mathbf{V}_n$ be a finite dimensional subspace of $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ and $\mathcal{P}_n$ be the projection from $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ onto $\mathbf{V}_n$. The Galerkin approximation of the operator equation $\mathcal{A}\sigma = g$ in $\mathbf{V}_n$ is given by $\mathcal{A}_n\sigma_n = g_n$ where

$\mathcal{A}_n = \mathcal{P}_n^*\mathcal{A}\mathcal{P}_n$, $\mathcal{P}_n^*$ is the adjoint operator of $\mathcal{P}_n$, $\sigma_n = \mathcal{P}_n\sigma$ and $g_n = \mathcal{P}_n^* g$. For any $v_n \in \mathbf{V}_n$, since $\mathcal{P}_n v_n = v_n$, we have

$$\langle \mathcal{A}_n v_n, v_n\rangle = \langle \mathcal{P}_n^*\mathcal{A}\mathcal{P}_n v_n, v_n\rangle = \langle \mathcal{A}\mathcal{P}_n v_n, \mathcal{P}_n v_n\rangle = \langle \mathcal{A} v_n, v_n\rangle.$$

From (11) and (12), we therefore get

$$\alpha\|v_n\|^2_{\mathbf{H}^{-\frac{1}{2}}} \le \langle \mathcal{A}_n v_n, v_n\rangle \le \beta\|v_n\|^2_{\mathbf{H}^{-\frac{1}{2}}}.$$

For the preconditioner $\mathcal{C}$, its Galerkin approximation in $\mathbf{V}_n$ is given by $\mathcal{C}_n = \mathcal{P}_n^*\mathcal{C}\mathcal{P}_n$. By using Theorem 2 and arguments similar to above, we also get, for all $v_n \in \mathbf{V}_n$

$$\alpha\|v_n\|^2_{\mathbf{H}^{-\frac{1}{2}}} \le \langle \mathcal{C}_n v_n, v_n\rangle \le \beta\|v_n\|^2_{\mathbf{H}^{-\frac{1}{2}}}.$$

Thus we see that for all $v_n \in \mathbf{V}_n$,

$$\frac{\alpha}{\beta} \le \frac{\langle \mathcal{A}_n v_n, v_n\rangle}{\langle \mathcal{C}_n v_n, v_n\rangle} \le \frac{\beta}{\alpha}.$$

Hence we have proved the following theorem.

Theorem 4 *The condition number of $\mathcal{C}_n^{-1}\mathcal{A}_n$ is of order $O(1)$ independent of n.*

Thus if the preconditioned conjugate gradient method is employed to solve the preconditioned system $\mathcal{C}_n^{-1}\mathcal{A}_n\sigma_n = \mathcal{C}_n^{-1}g_n$, then the convergence rate of the method is linear, see Golub and van Loan [11, Theorem 10.2.5].

Let us consider using the Galerkin method with piecewise polynomial basis functions $\{\psi_l\}_{l=1}^n$ on uniform grid to discretize the operators $\mathcal{A}$ and $\mathcal{C}$. By (13), the (k,l)th entry of the discrete matrix A_n of $\mathcal{A}$ will be given by

$$[A_n]_{k,l} = \int_0^{2\pi}\int_0^{2\pi} a(\theta,\phi)\psi_k(\theta)\psi_l(\phi)d\theta d\phi. \tag{25}$$

By (18) and (17), the (k,l)th entry of the Galerkin approximation C_n of $\mathcal{C}$ will be given by

$$[C_n]_{k,l} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} a(\zeta,\zeta-\theta+\phi)\psi_k(\theta)\psi_l(\phi)d\zeta d\theta d\phi.$$

Since $a(\cdot,\cdot)$ is 2π-periodic in both arguments, this becomes

$$[C_n]_{k,l} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} a(\zeta+\theta,\zeta+\phi)\psi_k(\theta)\psi_l(\phi)d\zeta d\theta d\phi. \tag{26}$$

We note that in general the integrations in (25) and (26) are difficult to evaluate. In the following, we prove that if the rectangle rule is used to discretize the inner-most integral in (26) (i.e. the one w.r.t. ζ), then the resulting matrix is equal to the *optimal circulant preconditioner* $c(A_n)$ of A_n, which is defined to be the minimizer of $\|B_n - A_n\|_F$ over all circulant matrices B_n, see T. Chan [2]. Here $\|\cdot\|_F$ is the Frobenius norm. Tyrtyshnikov [17] has shown that for a general matrix W_n, the entries of its optimal circulant preconditioner $c(W_n)$ are given by

$$[c(W_n)]_{k,l} = \frac{1}{n}\sum_{i-j=k-l(\mathrm{mod}\ n)} [W_n]_{i,j} = \frac{1}{n}\sum_{j=1}^n [W_n]_{(k+j)(\mathrm{mod}\ n),(l+j)(\mathrm{mod}\ n)}, \qquad 1 \le k,l \le n. \tag{27}$$

Theorem 5 *Let A_n be the Galerkin approximation of $\mathcal{A}$ as given by (25). Then the optimal circulant preconditioner $c(A_n)$ of A_n is equal to the Galerkin approximation of $\mathcal{C}$ as given in (26) with the inner-most integral (i.e. the one w.r.t. ζ) discretized by the rectangle rule.*

Proof: Using the rectangle rule, the (k,l)th entry of C_n in (26) becomes

$$
\begin{aligned}
& \frac{h}{2\pi}\sum_{j=1}^{n}\int_0^{2\pi}\int_0^{2\pi} a(jh+\theta, jh+\phi)\psi_k(\theta)\psi_l(\phi)d\theta d\phi \\
= & \frac{1}{n}\sum_{j=1}^{n}\int_0^{2\pi}\int_0^{2\pi} a(\theta,\phi)\psi_k(\theta - jh)\psi_l(\phi - jh)d\theta d\phi \\
= & \frac{1}{n}\sum_{j=1}^{n}\int_0^{2\pi}\int_0^{2\pi} a(\theta,\phi)\psi_{(k+j)(\mathrm{mod}\ n)}(\theta)\psi_{(l+j)(\mathrm{mod}\ n)}(\phi)d\theta d\phi \\
= & \frac{1}{n}\sum_{j=1}^{n}[A_n]_{(k+j)(\mathrm{mod}\ n),(l+j)(\mathrm{mod}\ n)}
\end{aligned}
$$

where the last equality follows from (25). Comparing this with (27), the theorem follows. □

In the numerical results in the next section, we will use the Galerkin method with piecewise constant polynomials to obtain the discrete approximation A_n to $\mathcal{A}$. In view of Theorem 5, we will use $c(A_n)$ as an approximation to $\mathcal{C}$ and then use it to precondition A_n. We note that if A_n has special structure, then $c(A_n)$ can be obtained fast. For example, if A_n is a Toeplitz matrix, i.e. it is constant along the diagonals, then $c(A_n)$ can be obtained in $O(n)$ operations, see (27). As another example, if A_n is a Hankel matrix, i.e. it is constant along the anti-diagonal, then $c(A_n)$ can still be obtained in $O(n)$ operations by summing the entries along the diagonals recursively, starting from the upper-right hand corner of the matrix.

5 Numerical Results

In this section, we illustrate the effectiveness of the circulant preconditioners by solving the boundary integral equation

$$\int_0^{2\pi} a(\theta,\phi)u(\phi)d\phi = g(\theta), \qquad 0 \le \theta \le 2\pi \tag{28}$$

on two types of regions: the ellipses and the dumb-bell shape regions. Since $\mathrm{diam}(\partial\Omega)$ may not be less than 1 for some of these regions, we have scaled $\partial\Omega$ such that (6) holds. More precisely, if $\delta = \mathrm{diam}(\partial\Omega)$ is the diameter of a given boundary, then we consider the scaled boundary $\partial\Omega_\rho = \{\rho x/\delta \mid x \in \partial\Omega\}$ with $0 < \rho < 1$. It is clear that $\mathrm{diam}(\partial\Omega_\rho) = \rho$. In our test, we have tested $\rho = 1/2$ and $\rho = 3/4$.

We discretize the equation and the right-hand side function in (28) by using the boundary element method with piecewise constant elements on uniform mesh. Since the basis elements are in $\mathbf{H}^{1/2-\epsilon}(\partial\Omega)$ for any $\epsilon > 0$, the computed solution $w(x)$ of (1) will have regularity $\mathbf{H}^{3/2-\epsilon}(\partial\Omega)$ on the boundary and $\mathbf{H}^{2-\epsilon}(\Omega)$ in Ω. This is generally acceptable as satisfactory, see for instance Chen and Zhou [9, p.312].

Using n elements, the resulting discretization matrix A_n is an n-by-n matrix. In view of Theorem 5, the circulant preconditioner is chosen to be $c(A_n)$. This amounts to integrating the inner-most integral in (26) by the rectangle rule. The system $A_n u_n = g_n$ is solved by using the preconditioned conjugate gradient method with or without the circulant preconditioner, see Golub and van Loan [11, p.529] for the algorithm of the method. In the computations, we have used the zero vector as the initial guess and the stopping criterion is $\|r_q\|_2/\|r_0\|_2 \le 10^{-10}$, where r_q is the residual vector at the qth iteration. All our computations were done in Matlab on an IBM 43P-133 workstation.

In the examples, in order to estimate the convergence of the computed solution u_n to the true solution u, we have used the right hand side function

$$g(\theta) = |\cos\theta|^{\frac{3}{2}}, \qquad 0 \le \theta \le 2\pi.$$

Clearly, $g(\theta) \in \mathbf{H}^{2-\epsilon}[0, 2\pi]$ and hence $u(\theta) \in \mathbf{H}^{1-\epsilon}[0, 2\pi]$, see for instance Hsiao and Wendland [13]. Therefore, u_n converges to u like $O(1/n)$ in $\mathbf{H}^0(\partial\Omega)$, see Hsiao and Wendland [14, Corollary 2.1] and also Chen and Zhou [9, p.316]. Because u is not known in closed form, a direct computation of the error is not possible. In the experiments, in order to illustrate the convergence rate, we have computed the relative error as follows:

$$e_n \equiv \frac{\|u_n - u_{n/2}\|_{\mathbf{H}^0}}{\|u_n\|_{\mathbf{H}^0}}. \tag{29}$$

Example 1. Here $\partial\Omega$ is the ellipse

$$x_1 = \mu\cos\theta, \quad x_2 = \nu\sin\theta, \qquad 0 \le \theta \le 2\pi.$$

For this problem, the kernel function (14) becomes

$$\begin{aligned} a(\theta,\phi) &= -\frac{1}{2\pi}\log\frac{\rho}{\delta}|2\sin(\frac{\theta-\phi}{2})| - \frac{1}{4\pi}\log(\mu^2\sin^2\frac{\theta+\phi}{2} + \nu^2\cos^2\frac{\theta+\phi}{2}) \\ &\equiv a_1(\theta,\phi) + a_2(\theta,\phi), \end{aligned}$$

where δ and ρ are respectively the diameters of the given and scaled boundaries. We note that $a_1(\theta,\phi)$ is a singular function whereas $a_2(\theta,\phi)$ is a smooth function. As mentioned above, the right hand side function is chosen to be $g(\theta) = |\cos\theta|^{\frac{3}{2}}$, $0 \le \theta \le 2\pi$.

To get the discretization matrix A_n from the Galerkin method, we need to compute

$$[A_n]_{kl} = \int_0^{2\pi}\int_0^{2\pi} a_1(\theta,\phi)\psi_k(\theta)\psi_l(\phi)d\theta d\phi + \int_0^{2\pi}\int_0^{2\pi} a_2(\theta,\phi)\psi_k(\theta)\psi_l(\phi)d\theta d\phi, \quad 1 \le k,l \le n, \tag{30}$$

where $\psi_l(\theta)$ is the piecewise constant function

$$\psi_l(\theta) = \begin{cases} 1/\sqrt{h} & \text{if } \theta \in [(l-1)h, lh), \\ 0 & \text{otherwise.} \end{cases} \qquad 1 \le l \le n,$$

see (25). Here $h = 2\pi/n$ is the mesh-size. Since $a_2(\theta,\phi)$ is smooth, we have computed the second integral in (30) over each element by the trapezoid rule with 3 points in each direction.

We note that the matrix it forms is a Hankel matrix, so only the first and the last columns are required. Therefore the matrix can be generated in $O(n)$ operations.

For the first integral in (30), since it forms a symmetric circulant matrix, only the first half of the first column is needed, i.e. integration is required on sub-intervals of $[0, \pi + h]$ for the first integral of (30). To evaluate the integrals, we write $a_1(\theta, \phi)$ as

$$a_1(\theta, \phi) = -\frac{1}{2\pi} \log \frac{\rho}{\delta} |\theta - \phi| - \frac{1}{2\pi} \log \left| \frac{2\sin(\frac{\theta-\phi}{2})}{\theta - \phi} \right|.$$

The integrals corresponding to the first term above can be evaluated analytically. Since the second term above has continuous second order derivative in $[0, \pi+h]$, its corresponding integrals on sub-intervals of $[0, \pi + h]$ can be evaluated by using the trapezoid rule again.

It follows from the above discussion that the discretization matrix A_n for this problem is a circulant-plus-Hankel matrix, which can be generated in $O(n)$ operations. Using the special structure of A_n, the matrix-vector multiplication of $A_n\mathbf{x}$ can be done in $O(n \log n)$ operations for any vector $\mathbf{x}$ by using the fast Fourier transform, see for instance Chan and Strang [8]. For the optimal circulant preconditioner $c(A_n)$, since it is a circulant matrix, the cost of multiplying $c(A_n)^{-1}$ to any vector can also be done in $O(n \log n)$ operations by using the fast Fourier transform, see [8]. Thus the cost per iteration of the conjugate gradient method with or without the circulant preconditioner is of $O(n \log n)$ operations.

We remark that the construction of the optimal circulant preconditioner $c(A_n)$ requires only $O(n)$ operations. This is because $c(\cdot)$ is a linear operator, see (27). Therefore $c(A_n)$ is equal to the sum of the optimal circulant preconditioner for the circulant part of A_n and the optimal circulant preconditioner for the Hankel part of A_n. Obviously, the optimal circulant preconditioner of any circulant matrix is the circulant matrix itself. The optimal circulant preconditioner of a Hankel matrix can be obtained in $O(n)$ operations by summing the entries along the diagonals recursively in (27), starting from the upper-right hand corner of the matrix.

Tables 1a and 1b give the number of iterations required for convergence for two different choices of ρ. We see from the tables that for the circulant-preconditioned systems, the numbers of iterations (listed under column $c(A_n)$) are fixed independent of n, as have been proven in Theorem 4. Hence the total cost of solving the system is of order $O(n \log n)$ operations. In contrast, the iteration numbers for the non-preconditioned systems (listed under column I) are increasing with respect to n. We also see that the relative error e_n as defined in (29) is decreasing like $O(1/n)$ as expected. We note that the convergence rates are independent of ρ, the diameter of the boundary $\partial\Omega$. Figure 1 gives the condition numbers of the matrix A_n and the preconditioned matrix $c(A_n)^{-1}A_n$ for $n = 32, 64, \ldots, 1024$ in log-log scale. We clearly see from the slope of the lines that the condition numbers are indeed of $O(n)$ and $O(1)$ respectively.

We remark that because of the scaling, the solution u of (28), which is in fact the solution $\sigma_1(y)$ in (4), will be different for different ρ. Therefore, the relative errors e_n in the tables are different for different ρ. However, the solution σ to (3) will be unique independent of the scaling as η and $\sigma_2(y)$ in (4) and (7) will be changed accordingly with the changing ρ.

Example 2. We consider dumb-bell shape curves (see Figure 2) defined in polar coordinates by

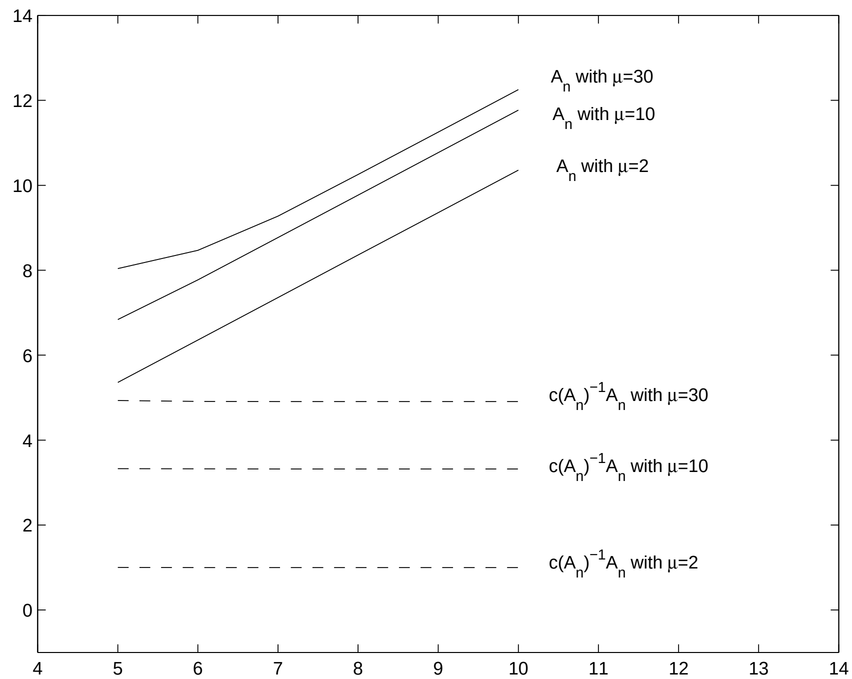
$$r = \cos 2\theta + f_\lambda(\theta), \qquad 0 \le \theta \le 2\pi,$$

$\rho = 1/2$	$\mu = 2,\ \nu = 1$			$\mu = 10,\ \nu = 1$			$\mu = 30,\ \nu = 1$		
n	I	$c(A_n)$	e_n	I	$c(A_n)$	e_n	I	$c(A_n)$	e_n
32	10	4	—	10	7	—	10	8	—
64	21	4	1.292e-1	20	8	1.486e-1	22	10	1.429e-1
128	32	4	6.710e-2	33	8	7.994e-2	35	10	7.856e-2
256	47	4	3.485e-2	44	8	4.257e-2	45	10	4.297e-2
512	61	4	1.807e-2	58	8	2.249e-2	61	10	2.313e-2
1024	79	4	9.347e-3	78	8	1.181e-2	84	10	1.230e-2
2048	106	4	4.826e-3	106	8	6.175e-3	106	10	6.484e-3

Table 1a: Numbers of Iterations for Different Ellipses.

$\rho = 3/4$	$\mu = 2,\ \nu = 1$			$\mu = 10,\ \nu = 1$			$\mu = 30,\ \nu = 1$		
n	I	$c(A_n)$	e_n	I	$c(A_n)$	e_n	I	$c(A_n)$	e_n
32	10	4	—	10	7	—	10	8	—
64	21	4	1.285e-1	21	8	1.483e-1	22	10	1.426e-1
128	31	4	6.671e-2	32	8	7.973e-2	34	10	7.843e-2
256	46	4	3.465e-2	44	8	4.246e-2	45	10	4.291e-2
512	61	4	1.796e-2	57	8	2.243e-2	60	10	2.309e-2
1024	79	4	9.293e-3	79	8	1.178e-2	80	10	1.228e-2
2048	106	4	4.798e-3	106	8	6.159e-3	106	10	6.474e-3

Table 1b: Numbers of Iterations for Different Ellipses.



solid line: condition numbers of A_n, dashed line: condition numbers of $c(A_n)^{-1}A_n$

Figure 1. Condition numbers (in log-log scale) of A_n and $c(A_n)^{-1}A_n$ for $\rho = 1/2$.

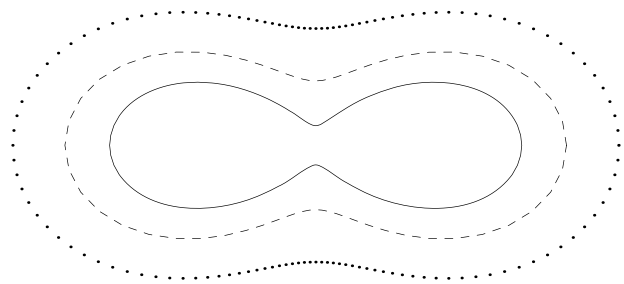
where $f_\lambda(\theta) = (\lambda^4 - \sin^2 2\theta)^{1/2}$ with $\lambda > 1$. After scaling and some straightforward computations, the kernel function (14) of these boundaries becomes

$$a(\theta,\phi) = -\frac{1}{2\pi}\log\frac{\rho}{\delta}|2\sin\frac{\theta-\phi}{2}| - \frac{1}{4\pi}\log\left\{4\sin^2(\theta+\phi)\cos^2(\frac{\theta-\phi}{2})\left(1+\frac{\cos 2\theta+\cos 2\phi}{f_\lambda(\theta)+f_\lambda(\phi)}\right)^2 + (\cos 2\theta + f_\lambda(\theta))(\cos 2\phi + f_\lambda(\phi))\right\}. \qquad (31)$$

The kernel function $a(\theta, \phi)$ can also be decomposed into two parts as in Example 1 and the discretization matrix A_n can be formed accordingly. Again, the right hand side is chosen to be $g(\theta) = |\cos\theta|^{\frac{3}{2}}$, $0 \le \theta \le 2\pi$.

For this problem, the discretization matrix A_n is a sum of two matrices in which the first term is a circulant matrix corresponding to the first term in (31). The second term is a dense matrix and has no special structure. Therefore the construction of A_n and the matrix-vector multiplication of A_n to any given vector both require $O(n^2)$ operations. However if fast multiplication schemes such as the fast multipole method are incorporated, see for instance [1, 6, 12], then the construction cost of A_n and the cost of the matrix-vector multiplication can both be reduced to $O(n)$ or $O(n\log n)$ operations.

The discretization matrix of $\mathcal{C}$ is again chosen to be $c(A_n)$. Since $c(A_n)$ is a circulant matrix, the multiplication of its inverse to any vector can be done in $O(n\log n)$ operations by using the fast Fourier transform. Since A_n is a dense matrix, construction of $c(A_n)$ requires $O(n^2)$ operations, see (27). However, in Chan, Lin, Ng and Sun [6], we have devised an algorithm to



solid line: $\lambda = 1.1$, dashed line: $\lambda = 1.3$, dotted line: $\lambda = 1.5$

Figure 2. The Curves of Different Dumb-Bell Shape Regions.

$\rho = 1/2$	$\lambda = 1.1$			$\lambda = 1.3$			$\lambda = 1.5$		
n	I	$c(A_n)$	e_n	I	$c(A_n)$	e_n	I	$c(A_n)$	e_n
32	9	6		9	5		9	5	
64	19	7	1.259e-1	20	6	1.198e-1	21	5	1.189e-1
128	29	7	6.547e-2	30	6	6.202e-2	30	5	6.339e-2
256	40	7	3.405e-2	41	6	3.213e-2	42	5	3.174e-2
512	55	7	1.768e-2	55	6	1.663e-2	54	5	1.759e-2
1024	71	7	9.157e-3	72	6	8.587e-3	74	5	8.452e-3
2048	93	7	4.733e-3	94	6	4.427e-3	95	5	4.351e-3

Table 2a: Numbers of Iterations for Dumb-Bell Shape Regions.

construct $c(A_n)$ in $O(n \log n)$ operations, see Table 9 in [6]. From Tables 2a and 2b, we see that the iteration numbers of the preconditioned systems, as listed in column $c(A_n)$ in the tables, are uniformly bounded whereas those of the original systems, listed in column I, are increasing with n as expected. Moreover, the relative error e_n is also decreasing like $O(1/n)$. In Figure 3, we also give in log-log scale the condition numbers of the matrix A_n and the preconditioned matrix $c(A_n)^{-1}A_n$ for $n = 32, 64, \ldots, 1024$. From the slopes of the lines, we see that the condition numbers are indeed of $O(n)$ and $O(1)$ respectively.

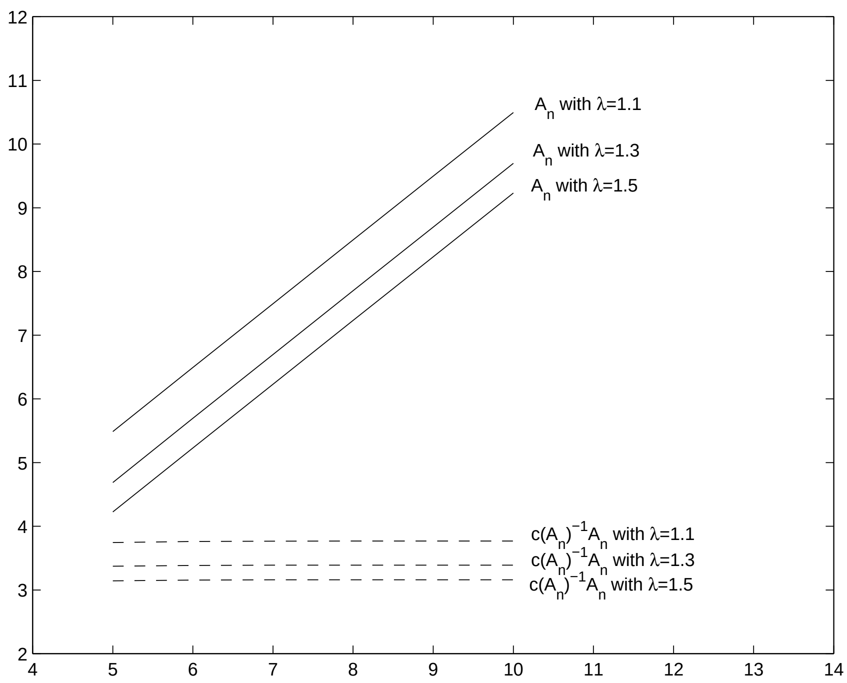
We emphasize again that if the fast multiplication scheme developed by Chan, Lin, Ng and Sun in [6] is used, then the construction of A_n, the matrix-vector multiplication of A_n to any vector, and the construction of $c(A_n)$ all require only $O(n \log n)$ operations, see Tables 7 and 9 in [6]. Thus the total cost of solving the system is of order $O(n \log n)$ operations if our fast multiplication scheme is employed.

6 Concluding Remarks

We have shown in this paper that optimal circulant integral operators can be used to precondition ill-conditioned matrices arising from boundary integral equations for the harmonic Dirichlet problem. The resulting preconditioned systems are well-conditioned and the convergence rate of our method is linear. We note that the results obtained here can be extended readily to other boundary integral equations as long as its bilinear form is positive definite and continuous (see

$\rho = 3/4$	$\lambda = 1.1$			$\lambda = 1.3$			$\lambda = 1.5$		
n	I	$c(A_n)$	e_n	I	$c(A_n)$	e_n	I	$c(A_n)$	e_n
32	9	6		9	6		9	5	
64	19	7	1.164e-1	20	6	1.134e-1	21	5	1.135e-1
128	29	7	6.030e-2	30	6	5.854e-2	31	5	5.850e-2
256	40	7	3.126e-2	41	6	3.025e-2	41	5	3.018e-2
512	54	7	1.618e-2	54	6	1.562e-2	53	5	1.556e-2
1024	70	7	8.362e-3	71	6	8.049e-3	74	5	8.009e-3
2048	94	7	4.313e-3	94	6	4.142e-3	95	5	4.117e-3

Table 2b: Numbers of Iterations for Dumb-Bell Shape Regions.



solid line: condition numbers of A_n, dashed line: condition numbers of $c(A_n)^{-1}A_n$

Figure 3: Condition numbers (in log-log scale) of A_n and $c(A_n)^{-1}A_n$ for $\rho = 1/2$.

Theorem 2).

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