On the Convergence Rate of a Newton-Like Method for Inverse Eigenvalue and Inverse Singular Value Problems

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Abstract: In this paper, we first note that Method III in Friedland, Nocedal, and Overton [*SIAM J. Numer. Anal., 24 (1987), pp. 634–667*] may not converge quadratically in the quotient sense. Then, we show that the method is convergent quadratically under a weaker notion of convergence — the root convergence. We also extend our results to the algorithm given in Chu [*SIAM J. Numer. Anal., 29 (1992), pp. 885–903*] for inverse singular value problems.

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1. INTRODUCTION

Let $\{A_i\}_{i=0}^n$ be real symmetric n-by-n matrices. For any vector $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T \in \mathbb{R}^n$, we define $A(\mathbf{c}) \equiv A_0 + \sum_{i=1}^n c_i A_i$. We denote the eigenvalues of $A(\mathbf{c})$ by $\{\lambda_i(\mathbf{c})\}_{i=1}^n$ with $\lambda_1(\mathbf{c}) \le \lambda_2(\mathbf{c}) \le \cdots \le \lambda_n(\mathbf{c})$. The inverse eigenvalue problem is defined as follows:

IEP: Given n real numbers $\{\lambda_i^*\}_{i=1}^n$, which are ordered as $\lambda_1^* \le \cdots \le \lambda_n^*$, find a vector $\mathbf{c}^* \in \mathbb{R}^n$ such that $\lambda_i(\mathbf{c}^*) = \lambda_i^*$ for $i = 1, \ldots, n$.

An IEP can be viewed as a problem of solving the nonlinear system of equations

$$\mathbf{f}(\mathbf{c}) = (\lambda_1(\mathbf{c}) - \lambda_1^*, \cdots, \lambda_n(\mathbf{c}) - \lambda_n^*)^T = \mathbf{0}. \tag{1}$$

Four numerical methods for solving (1) were given in the important paper by Friedland, Nocedal, and Overton [4]. They are related to Newton's method and have fast local convergence. Their methods have been widely used in many different applications, see for instance [2, 5, 7, 8, 9]. In particular, Method III in their paper was extended by Chu [1] to solve inverse singular value problems.

In this paper we will study the convergence rate of Method III in their paper in depth. The method is claimed to generate a sequence of iterates $\{\mathbf{c}_k\}$ converging to $\mathbf{c}^*$ quadratically in the quotient sense, i.e.

$$\|\mathbf{c}^{k+1} - \mathbf{c}^*\|_2 = O(\|\mathbf{c}^k - \mathbf{c}^*\|_2^2). \tag{2}$$

This claim is stated without an explicit proof in [4]. The aim of this paper is to investigate this claim. More precisely, we will point out that (2) may not hold for Method III without additional conditions.

Then we will show that a weaker convergence result, called the root-convergence or simply R-convergence (see [6, Chap. 9] or §4 for its definition), holds for Method III, and the R-convergence rate is at least 2. In contrast, the convergence claimed in (2) is called the quotient-convergence (or simply Q-convergence) with Q-convergence rate 2, see [6, Chap. 9]. We remark that the R-convergence rate of a sequence is always larger than or equals to its Q-convergence rate [6, p. 296]. Thus our results do not contradict the claim in [4]. Our aim here is intended to point out that the Q-quadratic convergence of Method III does not hold trivially and that a precise proof is needed.

The outline of the paper is as follows. In §2, we briefly review Method III. In §3, we give an example to show that one cannot derive (2) with the results that are explicitly proven in [4]. In §4, we show that the R-order of Method III is at least 2. In §5, we extend our results to the method given in [1] for inverse singular value problems. Concluding remarks are given in §6.

2. Method III

Method III is an iterative method based on Cayley transforms. Here we briefly recall it for the case where the given eigenvalues $\lambda_1^*, \cdots, \lambda_n^*$ are distinct. For details we refer to [4].

Let $\mathbf{c}^*$ be a solution to the IEP. There exists an orthogonal matrix Q^* such that

$$(Q^*)^T A(\mathbf{c}^*) Q^* = \Lambda^* \quad \text{and} \quad \Lambda^* = \mathrm{diag}(\lambda_1^*, \ldots, \lambda_n^*). \tag{3}$$

Suppose that $\mathbf{c}^k$ and the orthogonal matrix $Q^{(k)}$ are the current approximations of $\mathbf{c}^*$ and Q^*, respectively. Then, $Q^* = Q^{(k)} e^{Y_k}$ where Y_k is a skew-symmetric matrix and (3) can be written as

$$\begin{aligned}(Q^{(k)})^T A(\mathbf{c}^*) Q^{(k)} = e^{Y_k}\Lambda^* e^{-Y_k} &= (I + Y_k + \frac{1}{2}Y_k^2 + \cdots)\Lambda^*(I - Y_k + \frac{1}{2}Y_k^2 + \cdots) \\ &= \Lambda^* + Y_k\Lambda^* - \Lambda^* Y_k + O(\|Y_k\|^2),\end{aligned} \tag{4}$$

where we use $\|\cdot\|$ to denote both the matrix and the vector 2-norms. The iterate $\mathbf{c}^k$ is updated by neglecting the term $O(\|Y_k\|^2)$ in (4). Namely, $\mathbf{c}^{k+1}$ satisfies

$$(Q^{(k)})^T A(\mathbf{c}^{k+1}) Q^{(k)} = \Lambda^* + Y_k\Lambda^* - \Lambda^* Y_k, \tag{5}$$

and can be computed by equating the diagonal elements of (5), i.e.

$$(\mathbf{q}_i^k)^T A(\mathbf{c}^{k+1}) \mathbf{q}_i^k = \lambda_i^*, \quad i = 1, \ldots, n, \tag{6}$$

where $\{\mathbf{q}_i^k\}_{i=1}^n$ are the column vectors of $Q^{(k)}$. The n equations in (6) can be written as

$$J_k \mathbf{c}^{k+1} = \boldsymbol{\lambda}^* - \mathbf{b}^k, \tag{7}$$

where $\boldsymbol{\lambda}^* = (\lambda_1^*, \cdots, \lambda_n^*)^T$, $\mathbf{b}^k = ((\mathbf{q}_1^k)^T A_0 \mathbf{q}_1^k, \cdots, (\mathbf{q}_n^k)^T A_0 \mathbf{q}_n^k)^T$, and

$$[J_k]_{ij} = (\mathbf{q}_i^k)^T A_j \mathbf{q}_i^k, \quad i, j = 1, \ldots, n. \tag{8}$$

Once we get $\mathbf{c}^{k+1}$, the matrix Y_k is determined by equating the off-diagonal elements of (5), i.e.

$$[Y_k]_{ij} (\lambda_j^* - \lambda_i^*) = (\mathbf{q}_i^k)^T A(\mathbf{c}^{k+1}) \mathbf{q}_j^k, \quad 1 \le i < j \le n. \tag{9}$$

Finally, $Q^{(k)}$ is updated by setting $Q^{(k+1)} = Q^{(k)} P_k$ where P_k is an orthogonal matrix computed using the Cayley transform: $P_k = (I + \frac{1}{2} Y_k)(I - \frac{1}{2} Y_k)^{-1}$.

We summarize the algorithm as follows:

Algorithm I (Method III)

1. *Given* $\mathbf{c}^0$, *compute the exact eigenvectors* $\{\mathbf{q}_i(\mathbf{c}^0)\}_{i=1}^n$ *of* $A(\mathbf{c}^0)$. *Let*

$$Q^{(0)} = [\mathbf{q}_1, \ldots, \mathbf{q}_n] = [\mathbf{q}_1(\mathbf{c}^0), \ldots, \mathbf{q}_n(\mathbf{c}^0)].$$

2. *For k=0, 1, 2, ...*

   (a) *Form the matrix* J_k *by (8).*

   (b) *Solve* $\mathbf{c}^{k+1}$ *from (7).*

   (c) *Form the skew-symmetric matrix* Y_k *by (9).*

   (d) *Compute* $Q^{(k+1)} = [\mathbf{q}_1^{k+1}, \ldots, \mathbf{q}_n^{k+1}]$ *by solving*

$$(I + \frac{1}{2} Y_k)(Q^{(k+1)})^T = (I - \frac{1}{2} Y_k)(Q^{(k)})^T.$$

Notice that the Jacobian of the function $\mathbf{f}$ defined in (1) has the the form: $[J(\mathbf{c})]_{ij} = \mathbf{q}_i^T(\mathbf{c}) A_j \mathbf{q}_i(\mathbf{c})$, $i, j = 1, \ldots, n$, where $\{\mathbf{q}_i(\mathbf{c})\}_{i=1}^n$ are the exact eigenvectors of $A(\mathbf{c})$. Thus the matrix J_k given in (8) can be viewed as an approximation to $J(\mathbf{c}^k)$ and Method III is a Newton-like method.

3. Q-Convergence Rate of Method III

In [4], the error matrix $E^{(k)} \equiv Q^{(k)} - Q^*$ is shown to converge Q-quadratically to zero:

Theorem 1 [4, Theorem 3.4] *There exists a scalar* $\epsilon > 0$ *such that, if* $\|E^{(0)}\| \le \epsilon$, *then* $\{\|E^{(k)}\|\}$ *converges quadratically to zero.*

If one checks the proof of Theorem 3.4 in [4], one sees that it basically provided two relationships. One of them is the statement of the theorem, i.e.

$$\|E^{(k+1)}\| = O(\|E^{(k)}\|^2), \quad k = 0, 1, 2, \ldots, \tag{10}$$

see [4, Equation (3.65)]. The other is shown in the course of proving the theorem and it has the form

$$\|\mathbf{c}^{k+1} - \mathbf{c}^*\| = O(\|E^{(k)}\|^2), \quad k = 0, 1, 2, \ldots, \tag{11}$$

see [4, Equation (3.61)]. At the end of the proof of the theorem, it was stated that it is easy to modify the proof so as to show that $\{\mathbf{c}^k\}$ converges Q-quadratically to $\mathbf{c}^*$, i.e. (2) holds.

We believe that the Q-quadratic convergence of $\{\mathbf{c}^k\}$ cannot be established with only (10) and (11) alone, without additional conditions or proofs. Our reasons are as follows:

1. We first note that (2) does not follow from (10) and (11) alone. This can be shown by the following example. Consider

$$z_k \equiv \|E^{(k)}\| = \left(\frac{1}{8}\right)^{2^k}, \quad k = 0, 1, 2, \ldots,$$

and

$$u_k \equiv \|\mathbf{c}^k - \mathbf{c}^*\| = \begin{cases} \left(\frac{1}{16}\right)^{2^k} & \text{for } k \text{ even}, \\ \left(\frac{1}{8}\right)^{2^k} & \text{for } k \text{ odd}. \end{cases} \tag{12}$$

It is obvious that both z_k and u_k converge to 0 as $k \to \infty$. Also, for all k, $z_{k+1} = z_k^2$. Hence $\{z_k\} = \{\|E^{(k)}\|\}$ satisfies (10). Moreover, for k even, $u_{k+1} = z_{k+1} = z_k^2$, and for k odd, $u_{k+1} \leq z_k^2$. Therefore $\{u_k\} = \{\|\mathbf{c}^k - \mathbf{c}^*\|\}$ satisfies (11) too. However, for all k even,

$$\frac{u_{k+1}}{(u_k)^2} = \frac{\left(\frac{1}{8}\right)^{2^{k+1}}}{\left(\frac{1}{16}\right)^{2^k \cdot 2}} = \left(\frac{16}{8}\right)^{2^{k+1}} = 2^{2^{k+1}} \to +\infty, \quad \text{as } k \to \infty.$$

Hence, the Q-convergence rate (or Q-order) of $\{u_k\} = \{\|\mathbf{c}^k - \mathbf{c}^*\|\}$ is strictly less than 2. In fact, we can easily show that $u_{k+1} = O(u_k^{1.5})$ for all k, i.e. its Q-order is 1.5. From this example, we see that (10) and (11) alone cannot lead to (2).

2. We observe that in the proof of Theorem 3.4 in [4], there are no estimates of the form $\|\cdot\| = O(\|\mathbf{c}^k - \mathbf{c}^*\|^\beta)$ for any β, and they are needed in order to complete the proof of (2). For example, an additional relation like

$$\|E^{(k)}\| = O(\|\mathbf{c}^k - \mathbf{c}^*\|) \tag{13}$$

is sufficient to complete the proof. The proof of (13) for Method III is missing in [4]. However, for Method II in [4], this relation is proven explicitly [4, Equations (3.50) and (3.57)], and therefore one can establish the Q-quadratic convergence of Method II. We note that (13) cannot be derived trivially from (10) and (11) as the definition of $O(\cdot)$ is $O(w_k) \leq \alpha \cdot w_k$ and not $O(w_k) = \alpha \cdot w_k$.

3. We remark that one can obtain a necessary and sufficient condition for the Q-quadratic convergence in the form $\|\cdot\| = O(\|\mathbf{c}^k - \mathbf{c}^*\|^\beta)$ as follows. By the definition of $J(\mathbf{c}^k)$ and (8), one can prove that J_k is close to $J(\mathbf{c}^k)$ when $Q^{(k)}$ is close to Q^*. However, one cannot apply the standard Q-quadratic convergence analysis of Newton's method to Method III. This is because in Method III, we have $J(\mathbf{c}^k)(\mathbf{c}^{k+1} - \mathbf{c}^k) = -\mathbf{f}(\mathbf{c}^k) + \mathbf{r}_k$ where

$$\mathbf{r}_k = (J(\mathbf{c}^k) - J_k)\mathbf{c}^{k+1} + \mathbf{b}(\mathbf{c}^k) - \mathbf{b}^k \neq \mathbf{0},$$

$\mathbf{b}(\mathbf{c}^k) = (\mathbf{q}_1^T(\mathbf{c}^k)A_0\mathbf{q}_1(\mathbf{c}^k), \cdots, \mathbf{q}_n^T(\mathbf{c}^k)A_0\mathbf{q}_n(\mathbf{c}^k))^T$, and $\mathbf{b}^k$ is given in (7). Thus, Method III can be viewed as an inexact Newton method, see [3]. Since $J(\cdot)$ is Lipschitz continuous (cf. [4, Equation (3.24)]), by Lemma 3.1 of [3], $O(\|\mathbf{f}(\mathbf{c}^k)\|) = O(\|\mathbf{c}^k - \mathbf{c}^*\|)$. Hence by Theorem 3.3 of [3], one can conclude that Method III converges Q-quadratically if and only if $\|\mathbf{r}_k\| = O(\|\mathbf{c}^k - \mathbf{c}^*\|^2)$. However, it is not easy to see from the expression of $\mathbf{r}_k$ that the condition holds trivially.

4. Finally, we remark that one may proceed as in the proof of Method II in [4] and write

$$E^{(k)} = Q^{(k)} - Q(\mathbf{c}^k) + Q(\mathbf{c}^k) - Q^*,$$

where $Q(\mathbf{c}^k)$ is the matrix of the exact eigenvectors of $A(\mathbf{c}^k)$. The relation $\|Q(\mathbf{c}^k) - Q^*\| = O(\|\mathbf{c}^k - \mathbf{c}^*\|)$ is known to hold for $\mathbf{c}^k$ close to $\mathbf{c}^*$, see for instance [4, Equation (3.29)] and [9, p. 249]. However, we cannot bound $\|Q^{(k)} - Q(\mathbf{c}^k)\|$ by $O(\|\mathbf{c}^k - \mathbf{c}^*\|)$ as was done in the proof of Method II in [4]. The main reason is that in Method II, $Q^{(k)}$ is obtained by the inverse power method and is therefore related to $Q(\mathbf{c}^k)$. But in Method III, $Q^{(k)}$ is obtained via Cayley transforms and has no direct relationship with $Q(\mathbf{c}^k)$.

4. R-Convergence Rate of Method III

In the last section, we point out that it is not obvious from the proof of Theorem 3.4 of [4] that Method III converges Q-quadratically. In this section, however, we show that Method III converges quadratically under a weaker notion of convergence — the root-convergence. We first give its definition, see [6, Chap. 9].

Definition 1 *Let $\{\mathbf{x}^k\}$ be a sequence that converges to $\mathbf{x}^*$. Then the numbers*

$$R_p\{\mathbf{x}^k\} = \begin{cases} \limsup_{k\to\infty} \|\mathbf{x}^k - \mathbf{x}^*\|^{1/k}, & \text{if } p = 1, \\ \limsup_{k\to\infty} \|\mathbf{x}^k - \mathbf{x}^*\|^{1/p^k}, & \text{if } p > 1, \end{cases} \tag{14}$$

are the R-convergence factors of $\{\mathbf{x}^k\}$*. The quantity*

$$O_R(\mathbf{x}^*) = \begin{cases} \infty, & \text{if } R_p\{\mathbf{x}^k\} = 0, \forall p \in [1, \infty), \\ \inf\{p \in [1, \infty) \mid R_p\{\mathbf{x}^k\} = 1\}, & \text{otherwise}, \end{cases} \tag{15}$$

is called the R-convergence rate, or R-order, of $\{\mathbf{x}^k\}$*.*

We remark that the R-order is always larger than or equal to the Q-order, see [6, p. 296] (in fact, the R-order of $\{u_k\}$ in (12) is 2 while its Q-order is 1.5). It follows from the Q-quadratic convergence claim in Method III that the R-order of Method III is at least 2. Since it is not obvious that the Q-quadratic convergence claim of Method III is true, here we give a proof of the R-quadratic convergence of the method independent of the claim.

Theorem 2 *Under the assumptions of [4, Theorem 3.4],* $\{\mathbf{c}^k\}$ *converges to* $\mathbf{c}^*$ *with R-order at least* 2*.*

Proof: As mentioned, it was already proven in [4, Theorem 3.4] that (10) and (11) hold. Let $e_k \equiv \|E^{(k)}\|$. Then by (10) and (11), there exists a positive scalar $\sigma \geq 1$ such that

$$e_k \leq \sigma e_{k-1}^2 \quad \text{and} \quad \|\mathbf{c}^k - \mathbf{c}^*\| \leq \sigma e_{k-1}^2, \quad k = 1, 2, \ldots.$$

Hence, we have

$$\begin{aligned} \|\mathbf{c}^k - \mathbf{c}^*\| &\leq \sigma e_{k-1}^2 \leq \sigma \left(\sigma e_{k-2}^2\right)^2 = \sigma^{1+2} e_{k-2}^{2^2} \\ &\leq \sigma^{1+2} \left(\sigma e_{k-3}^2\right)^{2^2} = \sigma^{1+2+2^2} e_{k-3}^{2^3} \\ &\leq \cdots \leq \sigma^{1+2+2^2+\cdots+2^{k-1}} e_0^{2^k}. \end{aligned} \tag{16}$$

Since for any $k \geq 1$,

$$\frac{1 + 2 + 2^2 + \cdots + 2^{k-1}}{2^k} = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} \leq \frac{1/2}{1 - 1/2} = 1,$$

we see that (16) becomes

$$\|\mathbf{c}^k - \mathbf{c}^*\|_2 \leq \left(\sigma^{\frac{1+2+2^2+\cdots+2^{k-1}}{2^k}} e_0\right)^{2^k} \leq (\sigma e_0)^{2^k}. \tag{17}$$

Let $e_0 = \|E^{(0)}\|$ be sufficiently small such that $\sigma e_0 < 1$. From (17), we see that $\{\mathbf{c}^k\}$ converges to $\mathbf{c}^*$. Using (17), we now compute the root convergence factors of $\{\mathbf{c}^k\}$ for different values of p (see (14)):

1. If $p = 1$, then

$$R_1\{\mathbf{c}^k\} = \limsup_{k\to\infty} \|\mathbf{c}^k - \mathbf{c}^*\|_2^{1/k} \leq \limsup_{k\to\infty} (\sigma e_0)^{2^k/k} = 0.$$

2. If $1 < p < 2$, then

$$R_p\{\mathbf{c}^k\} = \limsup_{k\to\infty} \|\mathbf{c}^k - \mathbf{c}^*\|_2^{1/p^k} \le \limsup_{k\to\infty} (\sigma e_0)^{(2/p)^k} = 0.$$

3. If $p = 2$, then

$$R_2\{\mathbf{c}^k\} = \limsup_{k\to\infty} \|\mathbf{c}^k - \mathbf{c}^*\|_2^{1/2^k} \le \sigma e_0 < 1.$$

4. If $p > 2$, then

$$R_p\{\mathbf{c}^k\} = \limsup_{k\to\infty} \|\mathbf{c}^k - \mathbf{c}^*\|_2^{1/p^k} \le \limsup_{k\to\infty} (\sigma e_0)^{(2/p)^k} = 1.$$

Thus $R_p\{\mathbf{c}^k\} < 1$ for any $p \in [1, 2]$ and $R_p\{\mathbf{c}^k\} \le 1$ for any $p \in (2, \infty)$. Hence according to (15), $O_R(\mathbf{c}^*) \ge 2$. □

5. Extension to Inverse Singular Value Problems

In this section, we extend our results to the method given in Chu [1] for inverse singular value problems. We first state the problem.

ISVP: Given $\{B_i\}_{i=0}^n \subset \mathbb{C}^{m\times n}$ and n nonnegative real numbers $\sigma_1^* \ge \sigma_2^* \ge \cdots \ge \sigma_n^*$, find a vector $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T \in \mathbb{R}^n$ such that the singular values of the matrix $B(\mathbf{c}) \equiv B_0 + \sum_{j=1}^n c_j B_j$ are precisely $\sigma_1^*, \ldots, \sigma_n^*$.

Suppose $\mathbf{c}^*$ is an exact solution of the ISVP. Let $B(\mathbf{c}^*) = U\Sigma V^T$ be the singular value decomposition of $B(\mathbf{c}^*)$. Denote by $U^{(k)}$, $V^{(k)}$ and $\mathbf{c}^k$ the approximations of U, V and $\mathbf{c}^*$ as obtained by the method in [1]. Let $E^{(k)} \equiv \left(U^{(k)} - U, V^{(k)} - V\right)$ be the error matrix at the kth iteration. Then it was shown in [1, Theorem 4.2] that

$$\|E^{(k+1)}\|_F = O(\|E^{(k)}\|_F^2), \tag{18}$$

(see [1, Equation (71)]) and

$$\|\mathbf{c}^{k+1} - \mathbf{c}^*\| = O(\|E^{(k)}\|_F^2), \tag{19}$$

(see [1, Equation (82)]). Here $\|\cdot\|_F$ denotes the Frobenius norm.

As was shown by the example given in Item 1 in section 3, (18) and (19) alone are not sufficient to guarantee that $\{\mathbf{c}^k\}$ converges Q-quadratically. However, using the same technique as in section 4, we can easily conclude that the R-order of $\{\mathbf{c}^k\}$ is at least 2.

6. Concluding Remarks

In this paper, we point out that in [4] (respectively in [1]), only (10) and (11) (respectively only (18) and (19)) are explicitly proven. An example is given to show that these equations alone are not sufficient to prove the Q-quadratic convergence of their methods. We hope that our paper can motivate someone to give a precise Q-quadratic proof for the methods.

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