An uncertainty quantification framework for the achievability of backtesting results of trading strategies

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Abstract

Backtesting has always been an indispensable component in analyzing the profitability of trading strategies in empirical finance literature. When measuring return, while the majority of literature implicitly assumes that a trade can be implemented at the same closing price as the one generating the trading signal, others find empirical evidence suggesting that this assumption presents a significant challenge to the robustness of their results. Hence, several alternative return measurements have been proposed, including the incorporation of a one-day delay to mitigate this execution latency. The mixture of opinions regarding this issue triggers us to quantify the achievability of backtesting results in the presence of this implementation uncertainty. In particular, we propose a framework for implementing and backtesting trading strategies. A new concept called return at risk (RaR) is introduced to quantify such achievability, and we illustrate the proposed framework on a representative class of trading strategies. Results show that a significant number of technical trading strategies with positive returns are found to be not achievable in the presence of implementation uncertainty.

Keywords— Uncertainty Quantification; Quantitative Trading Strategies; Trading Strategies; Risk Management; Financial Simulation

1 Introduction

Backtesting is a key component in gauging the effectiveness of trading strategies. In essence, it provides various measures of the profitability of a trading strategy such as return by reproducing the strategy on historical data. While there is a myriad of studies on trading strategies, the vast majority of them uses only end-of-day market data for backtesting instead of intraday data due to its limited availability. An important question concerning the backtesting procedure remains: if a trading signal is generated upon observing a closing price, should the same closing price be used as the execution price? This issue is important as it directly affects the calculation of return which is at the core of profitability measurement.

However, literature on the profitability of trading rules is mixed regarding this issue. Some studies calculated trading returns by using the same closing prices as the ones generating the trading signals. In other words, an implicit assumption of these studies was that traders could perform transactions at the prices which were used to make decisions. Earlier papers include Fama and Blume (1966), Brock *et al.* (1992), LeBaron (1999), Knez and Ready (1996), Allen and Karjalainen (1999), and more recent papers include Zhu and Zhou (2009), Teixeira and De Oliveira (2010), Fang *et al.* (2014), Wang *et al.* (2015), Chan *et al.* (2016). In practice, however, if a trading decision is made based on the closing price, it is not feasible to execute the trade until the market reopens on the next day.

On the other hand, other studies recognized the infeasibility of such a transaction. Consequently, to ensure that there were no such "look-ahead" bias, a one-day delay was imposed by assuming that trades were executed at the closing price one day after a trading decision had been made. These studies include Lo *et al.* (2000), Ma *et al.* (2013). Similarly, some studies conducted a robustness test by comparing technical trading returns with and without a one-day delay. The studies by Sweeney (1988), Taylor (1992), Bessembinder and Chan (1995), Day and Wang (2002) documented a substantial decrease in portfolio excess returns upon incorporating a one-day delay. While imposing a one-day delay resolves the feasibility of execution, such a delay may distort the trading strategy in the sense that the trading signal after the one-day delay may deviate from the originally generated trading signal. This inconsistency may render the execution according to the original trading signal conflicting, and consequently such a robustness test may not have any implication on the trading strategy itself. Kozhan and Tham (2012) investigated the execution risk in high-frequency trading and demonstrated that the risk arises from the crowding effect of competing traders using arbitrage strategies. In light of the mixture of opinions regarding the above implementation uncertainty, we quantify in this paper the achievability of backtesting results in the presence of this implementation uncertainty by proposing an uncertainty quantification framework.

Uncertainty quantification was originally developed in the engineering community. It is used to seek bounds on a system's behavior such as the probability of system failure without full knowledge of the underlying probability structure. Recently, it has been applied to the field economics and finance as regulatory agencies place heavier emphasis on stress testing. In particular, uncertainty quantification frameworks have been proposed to test the soundness of financial portfolios. For example, in Chen *et al.* (2015), a new measure of fundamental economic uncertainty was developed from the yield curve in order to stress test a fixed-income portfolio. The measure is based on McDiarmid's distance and optimal uncertainty quantification methods.

In this paper, the proposed uncertainty quantification framework consists of a trading system and an accompanying backtesting system. The trading system is based on a trading strategy and a decision rule, while the backtesting system is based on a probability distribution of the return difference, and a newly defined concept called return at risk (RaR). We illustrate the proposed framework on a representative class of trading strategies found in Brock *et al.* (1992). Results show that a significant number of technical trading rules with positive returns are found to be not achievable when implementation uncertainty is taken into account.

The rest of the paper is organized as follows. Section 2 discusses the data assumption of the framework, which involves a stochastic model to be used later in both the trading and backtesting system. Section 3 describes the trading system, with trading strategy and decision rule as the core components of the system. The trading system is to be used alongside the backtesting system. Section 4 lays the groundwork for the backtesting system by describing methods to obtain the probability distribution of the return difference. Section 5 continues to discuss the backtesting system by proposing a new concept called return at risk (RaR) to quantify the achievability of backtesting results. Section 6 then illustrates the framework with a representative class of trading rules in Brock *et al.* (1992), and Section 7 concludes the paper.

2 Data Assumption

Before proceeding to describe the trading and backtesting system, it is worthwhile to note that both of them require a stochastic model of price. We model it with the geometric Brownian motion piecewisely for the trading hours of each trading day, and assume that the price dynamics of each trading day are independent of each other. We adopt this common model since Brownian motion has been well-studied, which makes the theoretical derivation of our framework easier. In the future,we will develop models that can be extended beyond the geometric Brownian motion to more general price models, and to models with multiple inputs such as price and volume. More precisely, we can choose another stochastic model to compute the joint probability distribution of the inputs, and the rest of the framework follows similarly.

To begin with, suppose the backtesting period is from day 0 to day N. From most major statistical databases, one can obtain the opening price O_i and closing price C_i for each trading day $i = 0, \dots, N$. Denote the intraday price of day i by S_t^i, where $t \in [0,1]$, so that S_0^i represents the opening price at day i, and S_1^i represents the closing price at day i. The above assumption that price movements of each trading day are independent implies

$$f\left(S_t^0, \dots, S_t^N \middle| O_0, \dots, O_N, C_0, \dots, C_N\right) = \prod_{i=0}^{N} f_i\left(S_t^i \middle| O_i, C_i\right),$$

where the function on the left-hand side is the joint probability distribution of the random variables $S_t^0, S_t^1 \ldots, S_t^N$ given $S_0^0 = O_0, S_0^1 = O_1, \ldots, S_t^N = O_N$, and $S_1^0 = C_0, S_1^1 = C_1, \ldots, S_1^N = C_N$, and all the functions on the right-hand side are the probability distribution of the random variables S_t^i given $S_0^i = O_i$ and $S_1^i = C_i$. With the above independence property, we can formulate the geometric Brownian motion for each day $i = 0, \ldots, N$ as follows: denoting the intraday drift by μ and intraday volatility by σ, we have

$$S_t^i = \left(S_0^i \exp\left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma B_t \right) \middle| S_0^i = O_i, S_1^i = C_i \right), \quad \text{for } t \in [0,1], \tag{1}$$

where B_t is the general Brownian bridge with $B_0 = 0$ and $B_1 = \frac{1}{\sigma}\left(\log\left(\frac{C_i}{O_i}\right) - \left(\mu - \frac{\sigma^2}{2}\right)\right)$.

To calibrate μ and σ, we simply calculate the mean and sample standard deviation of the log returns from the historical opening and closing prices of the backtesting period. More precisely, denote the historical opening and closing prices of the backtesting period by $O_0, \ldots, O_N$ and $C_0, \ldots, C_N$ respectively. Let $U_i = \log\left(\frac{C_i}{O_i}\right)$ for $i = 0, \ldots, N$ be the daily log returns. Here the opening price at day i is used instead of the closing price at day $i-1$ to avoid incorporating any discontinuity between the previous day's closing price and the subsequent day's closing price in the estimation of intraday drift and volatility. Then μ and σ can be estimated from the return series $U_0, \ldots, U_N$ by

$$\sigma^2 = \frac{1}{N} \sum_{i=0}^{N} \left(U_i - \bar{U}\right)^2 \text{ and } \mu = \bar{U} + \sigma^2/2,$$

where $\bar{U} = \frac{1}{N+1} \sum_{i=0}^{N} U_i$ is the mean of the return series. Notice that μ and σ are constant throughout the entire backtesting period, which are different from the μ_i's and σ_i's that will be introduced below. Once μ and σ are calibrated for the backtesting period, we use this model to evaluate the achievability of backtesting results of trading strategies.

3 Trading System

3.1 Trading Strategy

The first component of the trading system is the underlying trading strategy. Interday trading strategies of a security take various types of day-end data as inputs and return the desired position. We use integers to pinpoint the exact amount of securities being long (represented by a positive integer) or being short (represented by a negative integer). Since many well-known technical trading strategies take historical closing prices as the input only, we denote the closing price of day i (starting from $i = 0$) by $S_1^i = C_i$ as introduced in Section 2. In the future, we will develop models that can be extended to accommodate trading strategies that take input other than closing price, or even strategies that take more than one input (see Section 6.2 below) such as closing price, and trading volume. More precisely, instead of using the geometric Brownian motion to model the probability distribution of the closing price S_1^i (which will be illustrated in a moment), we may choose another suitable stochastic model to obtain the joint probability distribution of $\mathbf{d}_1^i$, a column vector containing the inputs. All the rest can then be proceeded in a similar manner. However, for notational convenience, we will restrict the description below to trading strategies that take closing price S_1^i as the only input.

Let $D_1^i = \begin{bmatrix} S_1^0 & S_1^1 & \cdots & S_1^i \end{bmatrix}$ be the $(i+1)$-vector containing the closing prices including and up to day i. We can describe the trading rule by a function as follows: let $g_i : \mathbb{R}_{>0}^{i+1} \longrightarrow \mathbb{Z}$ be given by $D_1^i \longmapsto Y_i$, where Y_i is the desired position taken at day i after observing all the closing prices including and up to day i, with positive Y_i's indicating long positions and negative Y_i's indicating short positions. Let $n_i = Y_i - Y_{i-1}$, the number of shares to be executed. A positive n_i represents buying n_i shares and a negative n_i represents short-selling n_i shares.

After the market closes on day $i-1$, all closing prices including and up to day $i-1$, i.e. $\{S_1^j\}_{j=0}^{i-1}$ are known. Therefore, we can use the previously defined g_i to define $\hat{g}_i : \mathbb{R}_{>0} \longrightarrow \mathbb{Z}$ by $S_1^i \longmapsto g_i\left(D_1^i\right) = Y_i$, which is a function taking the closing price of day i and returning the desired position. For each integer n_i, the execution region of n_i number of shares can then be defined by $R_{n_i} = \left\{S_1^i \in \mathbb{R}_{>0} : \hat{g}_i\left(S_1^i\right) - Y_{i-1} = n_i\right\}$. More precisely, R_{n_i} is a buy region if n_i is positive, a sell region if n_i is negative, and a no-action region if n_i is zero. Note that the execution regions constitute a partition of the whole domain $\mathbb{R}_{>0}$ in the sense that they are disjoint and exhaustive.

After obtaining the execution regions of trading strategies, the next step is to predict the closing price based on the price a short time δ before the market closes on day i. More precisely, we may obtain the probability distribution of S_1^i (closing price at day i) from $S_{1-\delta}^i$ (price a short time δ before the closing time of day i) by utilizing the geometric Brownian motion model described in Section 2. However, the μ and σ described above can only be found after the backtesting period, it cannot be known during real-time trading. For this reason, when trading real-time near the closing time of day i, we calibrate μ_i and σ_i from the most recent K days of historical data available at day i. In other words, we use $O_{i-K}, \ldots, O_{i-1}$ and $C_{i-K}, \ldots, C_{i-1}$ to form the log return series and calculate μ_i and σ_i using the same procedure above, so that

$$S_1^i = S_{1-\delta}^i \exp\left(\left(\mu_i - \frac{\sigma_i^2}{2}\right)\delta + \sigma_i B_\delta\right), \tag{2}$$

where B_t is the standard Brownian motion. Note that μ_i and σ_i are rolling, as the closing price is known up to day $i-1$ only when one is trading at day i. Clearly the expected closing price is

$$\mathbb{E}\left(S_1^i\right) = S_{1-\delta}^i \exp\left(\mu_i - \frac{\sigma_i^2}{2}\right)\delta\mathbb{E}\left(\exp\left(\sigma_i B_\delta\right)\right) = S_{1-\delta}^i \exp\left(\mu_i\delta\right). \tag{3}$$

Furthermore, we can construct the $(1-\alpha)\,100\%$ confidence interval of the closing price S_1^i as follows: taking the logarithm of (2), we have

$$\log\left(S_1^i\right) \sim \mathcal{N}\left(\log\left(S_{1-\delta}^i\right) + \left(\mu_i - \frac{\sigma_i^2}{2}\right)\delta, \sigma_i^2\delta\right),$$

where $\mathcal{N}\left(\mu, \sigma^2\right)$ denotes a normally distributed random variable with mean μ and variance σ^2. Therefore, the $(1-\alpha)\,100\%$ confidence interval of the closing price S_1^i on day i is given by

$$I_{\alpha,i} = \exp\left(\log\left(S_{1-\delta}^i\right) + \left(\mu_i - \frac{\sigma_i^2}{2}\right)\delta \pm z_\alpha\sigma_i\sqrt{\delta}\right),$$

where z_α is the critical value of the standard normal distribution for a confidence level of $1-\alpha$. Clearly, the probability $\mathbb{P}\left(I_{\alpha,i}\right) = 1-\alpha$. For convenience, we let L and U be the lower and upper bound of $I_{\alpha,i}$, so that

$$I_{\alpha,i} = (L, U). \tag{4}$$

3.2 Decision Rule

The second component of the trading system is a decision rule, which allows us to make a trading decision a short time δ before the market closes on day i. Once a trading decision is made, a market-on-close order can be placed before the market closes on day i. Admittedly, placing trade orders just before the market closes might potentially be suspectible to heightened scrutiny, as addressed in Hillion and Suominen (2004) and Comerton-Forde and Putnins (2011). Such market manipulation, however, is more prevalent in over-the-counter trades, for example as noted in Aggarwal and Wu (2006). The framework also alleviates the problem by avoiding placing trades directly at the market close but instead at an earlier time. Moreover, the potential for market manipulation may be lessened if we place the trades earlier by taking a larger δ.

With the above execution regions R_{n_i}, expected closing price $\mathbb{E}\left(S_1^i\right)$ and confidence interval $I_{\alpha,i}$, we consider the following common decision rules as samples for our study:

1. The first decision rule is to trade according to which execution region the expected closing price falls into. In other words, if $\mathbb{E}\left(S_1^i\right)$ falls into R_{n_i}, execution of n_i shares is decided.

2. The second decision rule is to trade according to the execution region which has the largest probability of containing S_1^i. In other words, we decide to execute n_i shares if $\mathbb{P}\left(R_{n_i}\right)$ is the largest among all execution regions.

3. The third decision rule utilizes the confidence interval $I_{\alpha,i}$ together with risk-averse behavior. Specifically, if $I_{\alpha,i} \cap R_{n_i} \neq \emptyset$ for some unique n_i, execution of n_i shares at the closing time can be planned. If there are multiple n_i's satisfying $I_{\alpha,i} \cap R_{n_i} \neq \emptyset$, this indicates that we are unsure about which execution region the closing price will fall into. Therefore, a risk-averse approach is to choose among the intersecting n_i's the one which minimizes the resulting exposure $|Y_i|$.

4. Finally, to complement the risk-averse behavior in the third decision rule, the fourth and final decision rule is essentially the same as the third one except that instead of adopting the risk-averse approach, we adopt a risk-loving approach by maximizing the resulting exposure $|Y_i|$.

Figures 1 to 4 illustrate the four decision rules in the case of three execution regions such that $R_{+1} = (M_i, \infty)$, $R_0 = [m_i, M_i]$, $R_{-1} = (0, m_i)$.

4 Backtesting System

As mentioned in Section 1, the backtesting system below is to be used alongside the trading system described in Section 3. Also, as intraday data are more expensive and less accessible, a practical benefit is added since the backtesting system uses only opening and closing prices, which are end-of-day data.

We recall some terminologies which will be used below. A *sample path* refers to a path of positions initiated according to the trading system. Since the backtesting system uses only opening and closing prices but the trading system makes use of the price $S_{1-\delta}^i$ a short time before the market closes, we need to simulate it according to the geometric Brownian motion model introduced in Section 2. This is the reason why randomness is involved and each sample path generated is different. Meanwhile, the *perfect path* refers to the path of positions initiated with the assumption that the closing prices can be used in both trading signal generation and execution. We emphasize again that this assumption cannot be accomplished in practice as it is not possible to conduct transactions after the market has closed. Finally, the *return difference* refers to the difference in return of a sample path with that of the perfect path. As there is randomness involved in a sample path, the return difference is regarded as a random variable denoted by $\mathfrak{D}$. The first step of the backtesting system is to obtain the probability distribution of the return difference $\mathfrak{D}$. There are two methods to accomplish it.

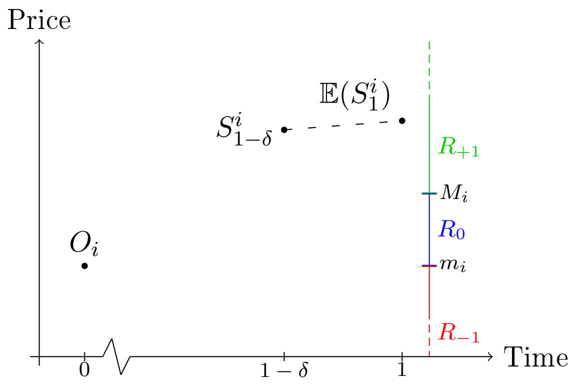

Figure 1 The figure illustrates the first decision rule. As the expected closing value of the stock price falls in R_{+1}, buying 1 share is decided.

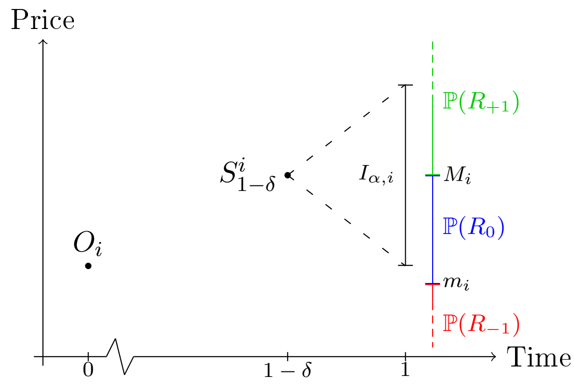

Figure 2 The figure illustrates the second decision rule. For example, if R_0 has the largest area under the probability distribution of S_1^i, we decide to do nothing.

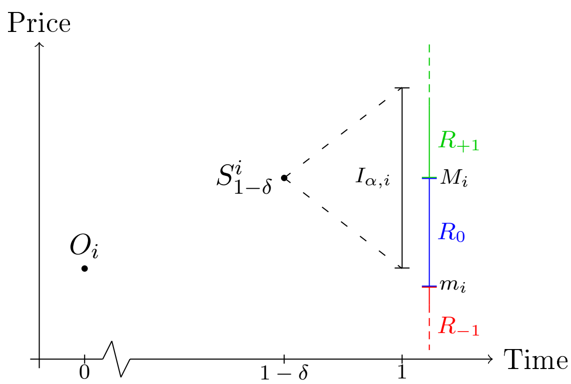

Figure 3 The figure illustrates the third decision rule. As the confidence interval intersects both R_{+1} and R_0, if the initial position held at time $i-1$ is -1, we decide to minimize our resultant exposure by buying 1 share.

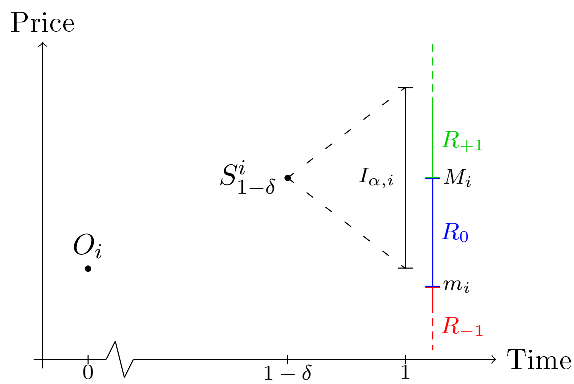

Figure 4 The figure illustrates the forth decision rule. As the confidence interval intersects both R_{+1} and R_0, if the initial position held at time $i-1$ is 0 (neutral), we decide to maximize our exposure by buying 1 share.

4.1 Probability Distribution of the Return Difference by Monte Carlo Simulation

The method of obtaining the probability distribution of return difference by Monte Carlo simulation is applicable to all four decision rules. In order to do the simulation, we first have to obtain the probability density function of the price $S^i_{1-\delta}$ a short period δ before the market closes for each day of the backtesting period. With σ calibrated from the entire backtesting period (described in Section 2), for each day $i = 1, \ldots, N$, we have the probability density function of $S^i_{1-\delta}$ given by

$$f_{S^i_{1-\delta}}\left(s^i_{1-\delta}\right) = \frac{1}{\sigma s^i_{1-\delta}\sqrt{2\pi\delta\left(1-\delta\right)}} \exp\left(-\frac{\left(\log\left(\frac{s^i_{1-\delta}}{O_i}\right) + \left(\delta - 1\right)\log\left(\frac{C_i}{O_i}\right)\right)^2}{2\sigma^2\delta\left(1-\delta\right)}\right), \tag{5}$$

for $s^i_{1-\delta} \in (0, \infty)$. The details of the derivation are given in appendix 8.1. We can then simulate a single value of $S^i_{1-\delta}$, for each day $i = 1, \ldots, N$ according to the above probability density function. After that, utilizing one of the four decision rules of our choice, we can determine the trading decision for each day $i = 1, \ldots, N$. Then we can calculate the return of this sample path from day 1 to day N by recording the closing prices corresponding to the trades. The return difference $\mathfrak{D}$ of this sample path with the perfect path can be calculated. By repeating the above procedure for a large number of sample paths, the probability distribution of the return difference $\mathfrak{D}$ can be obtained.

4.2 Probability Distribution of the Return Difference by Analytical Derivation

Among the four decision rules introduced in Section 3.2, this method of obtaining the probability distribution of return difference by analytical derivation is applicable only to the first decision rule. For simplicity, we describe the method for strategies with only 3 possible positions $+1$, 0, -1 for each day i with execution regions (M_i, ∞), $[m_i, M_i]$, $(0, m_i)$ respectively. The method can be extended in a similar manner to treat strategies with an arbitrary number of execution regions.

From (3), we have $\mathbb{E}\left(S^i_1\right) = S^i_{1-\delta}\exp\left(\mu_i\delta\right)$ with μ_i calibrated from the most recent K days of historical data available at day i. According to the first decision rule, positions of $+1, 0, -1$ corresponds to the cases which $\mathbb{E}\left(S^i_1\right)$ lies in (M_i, ∞), $[m_i, M_i]$, $(0, m_i)$ respectively. Equivalently, they correspond to the cases that $S^i_{1-\delta}$ lies in $\left(M_i e^{-\mu_i\delta}, \infty\right)$, $\left[m_i e^{-\mu_i\delta}, M_i e^{-\mu_i\delta}\right]$, $\left(0, m_i e^{-\mu_i\delta}\right)$ respectively. Subsequently, for each historical day $i = 1, \ldots, N$, we have the probabilities of each possible position as follows:

$$\begin{aligned}
\mathbb{P}_i\left(+1\right) &= \int_{M_i e^{-\mu_i\delta}}^{\infty} f_{S^i_{1-\delta}}\left(s^i_{1-\delta}\right) ds^i_{1-\delta}, \\
\mathbb{P}_i\left(0\right) &= \int_{m_i e^{-\mu_i\delta}}^{M_i e^{-\mu_i\delta}} f_{S^i_{1-\delta}}\left(s^i_{1-\delta}\right) ds^i_{1-\delta}, \\
\mathbb{P}_i\left(-1\right) &= \int_{0}^{m_i e^{-\mu_i\delta}} f_{S^i_{1-\delta}}\left(s^i_{1-\delta}\right) ds^i_{1-\delta}.
\end{aligned}$$

where $f_{S^i_{1-\delta}}$ is given by (5). Let $\left\{X_j\left(i\right) : j = 1, \ldots, 3^N\right\}$ be all the possible sample paths of the positions taken, where $X_j : \{1, \ldots, N\} \longrightarrow \{+1, 0, -1\}$. From historical closing prices, we can calculate the return difference $\mathfrak{D}$ of each sample path with that of the perfect path. The return difference of each sample path can then be associated with the probability of the path $\prod_{i=1}^{N} \mathbb{P}_i (X_j (i))$. Combining all such pairs, the probability distribution of the return difference $\mathfrak{D}$ can be obtained.

5 Return at Risk (RaR)

5.1 Definition of RaR

From the probability distribution of the return difference $\mathfrak{D}$, it would be nice to have a single value to quantify the risk of obtaining a return of a sample path lower than that of the perfect path (i.e. obtaining a negative return difference). To that end, we introduce the concept of *return ratio at risk* (RaR), which is analogous to the definition of the well-known value at risk (VaR) introduced by Artzner *et al.* (1999). It is defined as follows: given a confidence level β, we define the *return at risk* $\mathrm{RaR}_\beta (\mathfrak{D})$ to be the negative of level $(1-\beta)$-quantile of the return difference $\mathfrak{D}$, i.e.

$$\mathrm{RaR}_\beta (\mathfrak{D}) = -\inf \{ \mathfrak{d} \in \mathbb{R} : 1 - \beta \leq \mathbb{P} (\mathfrak{D} \leq \mathfrak{d}) \} .$$

5.2 Interpretation and Significance of RaR

Note that RaR reports the worst return difference not exceeded with a given level of confidence, regardless of whether the perfect return is positive or negative. For example, if the 95% RaR of a certain trading strategy with a decision rule is 0.2, then there is a 5% chance that the return from trading the strategy with the decision rule will be less than the perfect return by 0.2 of the absolute value of the perfect return. Conversely, if the 95% RaR of a certain trading strategy with a decision rule is -0.2, then there is a 95% chance that the return from trading the strategy with the decision rule will be more than the perfect return by 0.2 of the absolute value of the perfect return. A small positive or negative RaR means that a trading strategy with its decision rule is robust in the sense that up to the confidence level, a large fraction of the the return from backtesting is achievable in real-time trading. A negative RaR means that the return from trading real-time is even better than the return from backtesting up to the confidence level. Therefore in the extreme case, if the return from backtesting is negative but the RaR is also very negative (less than -1), then the strategy could still be launched. A large positive RaR, however, means that a large fraction of the return from backtesting is not achievable in the presence of implementation uncertainty; hence the strategy with its decision rule should not be launched even if the return from backtesting is appealing. The backtesting system and the use of RaR will be illustrated concretely in Section 6.4.

Traditionally, without the concept of RaR, a strategy will be deemed profitable and hence launched if the return from backtesting is attractive, without regarding to the fact that in practice one cannot use the closing price in both decision and execution. Hence the attractive return from backtesting may not be achievable in real trading. RaR thus gives additional information regarding the achievability of the return from backtesting in real trading, in the presence of implementation uncertainty. The additional information about trading strategies can also be used for more complete hypothesis testings. It will be interesting to see if RaR will give us a new perspective or conclusion when the framework is incorporated.

6 Illustration of the Framework

In this section, we present examples of applying the above framework to a representative set of trading strategies. As discussed in Section 1, there are two components of the framework. For decision rules, we have already given examples in Section 3.2. Therefore, it remains to provide examples of trading strategies to illustrate the framework.

6.1 Examples of Trading Strategies

The trading strategies chosen for illustration are from the representative set of twenty-six technical trading rules in Brock *et al.* (1992), which comprise of ten Variable Length Moving Average (VMA) rules, ten Fixed Length Moving Average (FMA) rules and six Trading Range Break (TRB) rules. Due to its popularity, we also include the Moving Average Convergence/Divergence (MACD) as introduced by Appel (2005) as well. An introduction to the above technical trading rules is given in Chan *et al.* (2014), and we adopt notations similar to them below.

We compute the execution regions for each of the above strategies. Note that VMA and MACD take only one type of day-end data, the closing price, as the input.

6.1.1 Execution Regions for VMA

Denote the L-period simple moving average at day i by $\mathrm{SMA}_{L,i}$, that is,

$$\mathrm{SMA}_{L,i} = \frac{1}{L} \sum_{j=i-L+1}^{i} S_1^j.$$

Using the moving average crossover rule with $L_1 < L_2$ and a $p\%$ band, the associated function $g_i : \mathbb{R}_{>0}^{i+1} \longrightarrow \mathbb{Z}$ is given by

$$g_i\left(\left[S_1^0, \ldots, S_1^{i-1}, S_1^i\right]\right) = \begin{cases} +1, & \text{if } \mathrm{SMA}_{L_1,i} > \mathrm{SMA}_{L_2,i} \times \left(1 + \frac{p}{100}\right), \\ -1, & \text{if } \mathrm{SMA}_{L_1,i} < \mathrm{SMA}_{L_2,i} \times \left(1 - \frac{p}{100}\right), \\ 0, & \text{otherwise.} \end{cases}$$

Define $\hat{g}_i$ accordingly. We further let

$$M_i = \frac{L_1\left(1 + \frac{p}{100}\right)}{L_2 - L_1\left(1 + \frac{p}{100}\right)} \sum_{j=i-L_2+1}^{i-L_1-1} S_1^j - \sum_{j=i-L_1+1}^{i-1} S_1^j, \tag{6a}$$

$$m_i = \frac{L_1\left(1 - \frac{p}{100}\right)}{L_2 - L_1\left(1 - \frac{p}{100}\right)} \sum_{j=i-L_2+1}^{i-L_1-1} S_1^j - \sum_{j=i-L_1+1}^{i-1} S_1^j. \tag{6b}$$

For $Y_{i-1} = 0$ (i.e. if no position are taken initially at day $i-1$), the execution regions are found to be

$$R_{+1} = (M_i, \infty), \tag{7a}$$

$$R_{-1} = (0, m_i), \tag{7b}$$

$$R_0 = [m_i, M_i]. \tag{7c}$$

Details for the derivation of (7) can be found in appendix 8.2. Similar results can be obtained for $Y_{i-1} = 1$ (where the above regions are changed to R_0, R_{-2}, R_{-1} accordingly) and $Y_{i-1} = -1$ (where the above regions are changed to R_{+2}, R_0, R_{+1} accordingly).

6.1.2 Execution Regions for FMA

The execution regions for FMA are similar to those for VMA, except that an extra dimension is used to capture a fixed holding period of H days once a long or short position is initiated. Let $W_1^i = \max\{w : Y_{i-1} = Y_{i-2} = \cdots = Y_{i-w} = 0\}$, the largest number of days for which no position is taken since then. Also, let M_i and m_i be the same as those from (6). Different from VMA and MACD, the function $\hat{g}_i$ of FMA takes two inputs S_1^i and W_1^i. For $Y_{i-1} = 0$, the execution regions are two-dimensional as follows:

$$R_{+1} = (M_i, \infty) \times (H, \infty)\,, \tag{8a}$$
$$R_{-1} = (0, m_i) \times (H, \infty)\,, \tag{8b}$$
$$R_0 = \mathbb{R}_{>0} \times \mathbb{Z}_{\geq 0} \backslash (R_{+1} \cup R_{\text{-}1})\,. \tag{8c}$$

For $Y_{i-1} = 1$, the execution regions are

$$R_{+1} = \emptyset, \tag{9a}$$
$$R_{-1} = \mathbb{R}_{>0} \times (H, \infty)\,, \tag{9b}$$
$$R_0 = \mathbb{R}_{>0} \times \mathbb{Z}_{\geq 0} \backslash R_{\text{-}1}. \tag{9c}$$

Similar results can be found for $Y_{i-1} = -1$.

6.1.3 Execution Regions for TRB

As for the Trading Range Break-out (TRB), we denote the L-day local maximum and minimum by $M_i = \max\left\{S_1^j : j = i - L, \ldots, i-1\right\}$ and $m_i = \min\left\{S_1^j : j = i - L, \ldots, i-1\right\}$ respectively. Then for $Y_{i-1} = 0$ and $Y_{i-1} = 1$, the execution regions are the same as (8) and (9), with M_i and m_i replaced by the above local maximum and minimum respectively.

6.1.4 Execution Regions for MACD

For MACD, we use $L_1 = 12$ and $L_2 = 26$ as the periods for the short and long term exponential moving average respectively. We let

$$M_i = \frac{(L_1+1)(L_2+1)}{2(L_2 - L_1)} \sum_{k=1}^{\infty} \left(\frac{2}{L_2+1} \left(\frac{L_2 - 1}{L_2 + 1} \right)^k - \frac{2}{L_1+1} \left(\frac{L_1 - 1}{L_1 + 1} \right)^k \right) S_1^{i-k}.$$

For $Y_{i-1} = 0$, the execution regions are

$$R_{+1} = (M_i, \infty)\,, \tag{10a}$$
$$R_{-1} = (0, M_i)\,, \tag{10b}$$
$$R_0 = \{M_i\}\,. \tag{10c}$$

Appendix 8.3 contains the details of the derivation.

6.2 Examples of Expected Closing Value and Confidence Regions

Note that for VMA and MACD, only the closing price is taken as the input; therefore the expected closing price and confidence interval is found by (3) and (4) respectively in Section 3.1. As for FMA and TRB, the associated functions take two types of data as the input, the closing price S_1^i and

Table 1 Illustration of the Trading System for VMA with the third decision rule

Inequality Satisfied	Intersecting Regions	Minimizing Exposure?	Trading Decision
$m_i < M_i < L < U$	R_{+1}	No	Long 1 share
$m_i < L < U < M_i$	R_0	No	No action
$L < U < m_i < M_i$	R_{-1}	No	Short 1 share
$m_i < L < M_i < U$	R_{+1}, R_0	Yes	No action
$L < m_i < U < M_i$	R_{-1}, R_0	Yes	No action
$L < m_i < M_i < U$	R_{+1}, R_{-1}, R_0	Yes	No action

the largest number of days for which no position is taken since then, i.e. W_1^i. Note that unlike S_1^i which cannot be determined until the closing time of day i, W_1^i is already known after the closing time of day $i-1$. Therefore, the expected closing value is

$$\mathbb{E}\left(\begin{bmatrix} S_1^i \\ W_1^i \end{bmatrix}\right) = \begin{bmatrix} S_{1-\delta}^i \exp(\mu_i \delta) \\ W_1^i \end{bmatrix}.$$

For the confidence region, it is only necessary to use the confidence interval of S_1^i in (4) (here denoted by $\tilde{I}_{\alpha,i}$), and then take the direct product with $\{W_1^i\}$ to give the confidence region $I_{\alpha,i} = \tilde{I}_{\alpha,i} \times \{W_1^i\}$, which is in fact a line segment in a two dimensional plane.

6.3 Illustration of the Trading System

With the above execution regions, expected closing values and confidence regions, we can make a trading decision a short time δ before the market closes. For example, suppose we are trading the VMA using the third decision rule (risk-averse behavior). For $Y_{i-1} = 0$, utilizing the execution regions in (7) and confidence interval defined in (4), the decision-making process can be illustrated by table 1.

With the above decision in mind, it remains to execute the trade by placing a guaranteed maket-on-close order to ensure execution at the closing price if such an order is available in the exchange. In Section 7, we will mention briefly how to incorporate other execution algorithms, especially when large orders cause market impact. Other combinations of trading strategies and decision rules can be performed similarly.

6.4 Illustration of the Backtesting System

6.4.1 Data and Results

In this section, we illustrate the backtesting system with the 27 technical trading rules introduced in Section 6.1 (VMA, FMA, TRB, MACD). The data series are the daily opening and closing values of the Dow Jones Industrial Average (DJIA) from September 2005 to August 2015. The 10-year daily data is extracted from Bloomberg Terminal provided by Bloomberg L.P. Since the New York Stock Exchange (NYSE) has normal trading hours from 9:30 a.m. to 4:00 p.m. Eastern Time, there are 390 minutes in a trading day. Using the previous notation, if we make a trading decision 10 minutes before the exchange closes, we have $\delta = 10/390$. Also, as explained in Section 3.1, when reproducing trades executed at day i of the backtesting period, we need to use a rolling μ_i and σ_i computed from the most recent K days available. For the illustration below, we take $K = 30$. Finally, we simulate $10,000$ sample paths for each combination of trading strategy and decision rule to construct the probability distribution of the return difference. Figure 5 shows an example of the

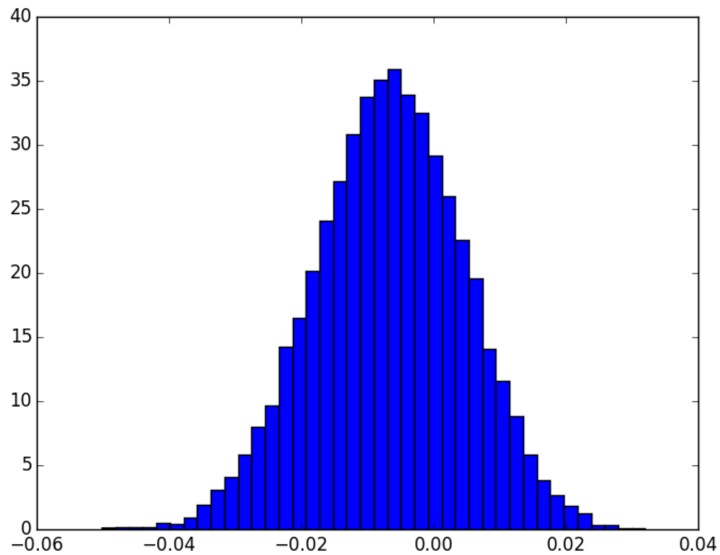

Figure 5 The probability distribution of the return difference of VMA with 1 as the short period, 50 as the long period, 0% band and the second decision rule from September 2005 to to August 2015.

probability distribution of the return difference of VMA with 1 as the short period, 50 as the long period, 0% band and the second decision rule. From the probability distribution, we can obtain the 95%, 99%, 99.5% and 99.9% return at risk. As noted in Harvey and Liu (2014), Bailey and López de Prado (2014), and Charles-Cadogan (2015), we also conduct a rudimentary performance evaluation of the trading strategies by including their Sharpe ratios as well. Tables 2 to 4 present the backtesting results.

6.4.2 Discussion

From the backtesting results in table 2 to 4, several outcomes are worthwhile to note. Firstly, it can be observed that for each combination of trading strategy and decision rule, RaR increases as the confidence level increases. This is a direct result of the definition of RaR, and matches the intuition that the return at risk is larger when a higher level of confidence is required.

Secondly, a significant number of strategies with positive perfect returns are found to be not achievable when implementation uncertainty is taken into account. For example, for FMA with 1 as the short period, 50 as the long period, 0% band and the third decision rule, the perfect return is 0.05082. However, the RaR at 95% confidence is 0.066428, which is larger than the perfect return by 0.015608. Subsequently, there is a 5% chance that this combination would suffer from a return worse than -0.015608 when implementation uncertainty is taken into account.

On the other hand, some strategies with positive returns are achievable even when implementation uncertainty is taken into account. For example, for FMA with 1 as the short period, 150 as the long period, 0% band and the forth decision rule, the perfect return is 0.07005 and the RaR at 95% confidence is 0.004398. Therefore, even when implementation uncertainty is taken into account, there is a 95% chance that this combination would enjoy a return of more than 0.065652, which is the difference between the perfect return and the 95% RaR.

Finally, some of the values of RaR above are 0, such as FMA with 5 as the short period, 150 as the long period, 0% band and the first decision rule. This means there is no return at risk at 95% confidence for this combination when implementation uncertainty is taken into consideration.

Table 2 Backtesting Results for Variable-Length Moving (VMA) Rules

The sample period is from September 2005 to to August 2015. The symbol $(L_1, L_2, p\%)$ is used to represent a VMA crossover strategy, where L_1 is the duration of the short moving average, L_2 is the duration of the long moving average, and a $p\%$ band is used to envelop the long moving average. PR refers to the perfect return. Sharpe refers to the Sharpe ratio, with the Federal funds rate as the risk-free rate. DR 1 to DR 4 represent the first to the forth decision rules. $100\beta\%$ RaR is the return at risk at a confidence level of β. All returns and Shapre ratios are calculated in logarithms and reported in annual scale.

Strategy	PR	Sharpe	RaR	DR 1	DR 2	DR 3	DR 4
(1, 50, 0%)	-0.011960	-0.000270	95% RaR	0.025718	0.025427	0.031193	0.031959
			99% RaR	0.033065	0.032829	0.035895	0.038572
			99.5% RaR	0.035524	0.035481	0.037825	0.041166
			99.9% RaR	0.042488	0.040238	0.040746	0.046724
(1, 50, 1%)	-0.017030	-0.000480	95% RaR	0.011505	0.011400	-0.000549	0.031911
			99% RaR	0.015557	0.016168	0.003858	0.036767
			99.5% RaR	0.017097	0.017690	0.005550	0.038766
			99.9% RaR	0.019989	0.020406	0.008828	0.043050
(1, 150, 0%)	0.019540	0.000160	95% RaR	0.007022	0.007114	0.000360	-0.001024
			99% RaR	0.011714	0.011857	0.003704	0.003857
			99.5% RaR	0.013551	0.013590	0.004641	0.005569
			99.9% RaR	0.017060	0.017044	0.007015	0.008817
(1, 150, 1%)	0.024210	0.000320	95% RaR	0.005513	0.005444	0.011961	0.005054
			99% RaR	0.008983	0.008802	0.015336	0.008719
			99.5% RaR	0.009868	0.010094	0.016605	0.009939
			99.9% RaR	0.012192	0.012954	0.018576	0.012238
(5, 150, 0%)	0.022240	0.000290	95% RaR	0.002240	0.002239	0.001666	0.001876
			99% RaR	0.003568	0.003470	0.002929	0.003478
			99.5% RaR	0.004041	0.004041	0.003445	0.003937
			99.9% RaR	0.004957	0.005233	0.004193	0.004915
(5, 150, 1%)	0.032360	0.000770	95% RaR	0.001114	0.001083	-0.003527	0.001475
			99% RaR	0.002133	0.002156	-0.001606	0.002936
			99.5% RaR	0.002517	0.002535	-0.000959	0.003609
			99.9% RaR	0.003162	0.003198	0.000211	0.005071
(1, 200, 0%)	0.056420	0.000970	95% RaR	0.015492	0.015226	0.023770	0.041293
			99% RaR	0.020129	0.020083	0.026572	0.045610
			99.5% RaR	0.022120	0.021848	0.027507	0.046677
			99.9% RaR	0.026104	0.025118	0.029222	0.049180
(1, 200, 1%)	0.029890	0.000530	95% RaR	0.004468	0.004409	0.002907	0.002969
			99% RaR	0.007245	0.007034	0.006234	0.006441
			99.5% RaR	0.008326	0.007922	0.007439	0.007702
			99.9% RaR	0.010446	0.010110	0.009571	0.009628
(2, 200, 0%)	0.050720	0.000980	95% RaR	0.014344	0.014389	0.015046	0.015287
			99% RaR	0.018137	0.017994	0.017300	0.018943
			99.5% RaR	0.019569	0.019239	0.017970	0.020426
			99.9% RaR	0.022368	0.021576	0.019500	0.023724
(2, 200, 1%)	0.023750	0.000420	95% RaR	0.003424	0.003491	-0.006925	0.002206
			99% RaR	0.005191	0.005378	-0.004703	0.003957
			99.5% RaR	0.005768	0.006110	-0.003928	0.004601
			99.9% RaR	0.007095	0.007405	-0.002370	0.005774

Table 3 Backtesting Results for Fixed-Length Moving (FMA) Rules

The sample period is from September 2005 to to August 2015. The symbol $(L_1, L_2, p\%)$ is used to represent a FMA crossover strategy, where L_1 is the duration of the short moving average, L_2 is the duration of the long moving average, and a $p\%$ band is used to envelop the long moving average. The fixed holding period H is set to be 10 days as suggested by Brock *et al.* (1992). PR refers to the perfect return. Sharpe refers to the Sharpe ratio, with the Federal funds rate as the risk-free rate. DR 1 to DR 4 represent the first to the forth decision rules. $100\beta\%$ RaR is the return at risk at a confidence level of β. All returns and Shapre ratios are calculated in logarithms and reported in annual scale.

Strategy	PR	Sharpe	RaR	Sharpe DR 1	DR 2	DR 3	DR 4
(1, 50, 0%)	0.050820	0.000640	95% RaR	0.014341	0.014172	0.066428	0.018488
			99% RaR	0.019621	0.019621	0.075472	0.021245
			99.5% RaR	0.020365	0.021335	0.078950	0.023787
			99.9% RaR	0.023600	0.023912	0.085578	0.025629
(1, 50, 1%)	-0.003090	-0.000210	95% RaR	0.014789	0.014592	-0.000233	0.022415
			99% RaR	0.021411	0.021575	0.009172	0.031023
			99.5% RaR	0.023857	0.024038	0.011947	0.034471
			99.9% RaR	0.030448	0.030267	0.019208	0.042248
(1, 150, 0%)	0.070050	0.000970	95% RaR	0.006424	0.006407	0.059451	0.004398
			99% RaR	0.010109	0.010607	0.072186	0.007425
			99.5% RaR	0.012519	0.012093	0.076219	0.007425
			99.9% RaR	0.015924	0.015859	0.084383	0.011629
(1, 150, 1%)	0.056850	0.000760	95% RaR	0.037173	0.037654	0.056559	0.022057
			99% RaR	0.044622	0.045473	0.063023	0.031516
			99.5% RaR	0.046962	0.047885	0.065215	0.034474
			99.9% RaR	0.051664	0.052749	0.068714	0.050006
(5, 150, 0%)	0.059310	0.000790	95% RaR	0.000000	0.000000	0.003482	0.001129
			99% RaR	0.001129	0.001129	0.004109	0.001129
			99.5% RaR	0.001129	0.001129	0.011142	0.001129
			99.9% RaR	0.001129	0.001129	0.043232	0.001129
(5, 150, 1%)	0.035860	0.000420	95% RaR	0.000000	0.000000	0.000190	0.001384
			99% RaR	0.000000	0.000000	0.003114	0.008436
			99.5% RaR	0.000000	0.000000	0.003707	0.020627
			99.9% RaR	0.001588	0.001998	0.004704	0.023465
(1, 200, 0%)	0.031390	0.000350	95% RaR	0.009264	0.009452	0.030078	0.024751
			99% RaR	0.012691	0.012691	0.035873	0.029853
			99.5% RaR	0.014897	0.014899	0.036864	0.029853
			99.9% RaR	0.016878	0.016878	0.045423	0.029853
(1, 200, 1%)	0.025900	0.000260	95% RaR	0.014265	0.013945	-0.013787	0.015534
			99% RaR	0.021952	0.020706	0.002925	0.024182
			99.5% RaR	0.024310	0.022926	0.006110	0.027110
			99.9% RaR	0.027928	0.028257	0.013136	0.032174
(2, 200, 0%)	0.026120	0.000260	95% RaR	0.008549	0.008549	0.027894	0.019920
			99% RaR	0.011633	0.011871	0.031347	0.019920
			99.5% RaR	0.014358	0.014358	0.031347	0.019920
			99.9% RaR	0.016033	0.017267	0.038558	0.023964
(2, 200, 1%)	0.018210	0.000130	95% RaR	0.005925	0.006286	0.005670	0.005130
			99% RaR	0.014279	0.014919	0.014988	0.010572
			99.5% RaR	0.016379	0.018985	0.017224	0.011729
			99.9% RaR	0.020074	0.022149	0.021791	0.014877

Table 4 Backtesting Results for Trading Range Break (TRB) Rules and MACD

The sample period is from September 2005 to to August 2015. The symbol $(L, p\%)$ is used to represent a TRB strategy, where L is the number of days taken for the local maximum and minimum and a $p\%$ band is used to envelop the local maximum and minimum. The fixed holding period H for each TRB strategy is set to be 10 days as suggested by Brock *et al.* (1992). PR refers to the perfect return. Sharpe refers to the Sharpe ratio, with the Federal funds rate as the risk-free rate. DR 1 to DR 4 represent the first to the forth decision rules. $100\beta\%$ RaR is the return at risk at a confidence level of β. All returns and Shapre ratios are calculated in logarithms and reported in annual scale.

Strategy	PR	Sharpe	RaR	DR 1	DR 2	DR 3	DR 4
(50, 0%)	-0.027200	-0.000810	95% RaR	0.003397	0.003489	0.013389	0.008073
			99% RaR	0.012278	0.012745	0.022299	0.015248
			99.5% RaR	0.015511	0.015877	0.026454	0.018058
			99.9% RaR	0.021042	0.022229	0.032799	0.022159
(50, 1%)	-0.030940	-0.001330	95% RaR	0.007986	0.007747	0.021898	-0.000607
			99% RaR	0.012444	0.012488	0.029239	0.005073
			99.5% RaR	0.014040	0.014584	0.031769	0.006580
			99.9% RaR	0.017630	0.018721	0.036084	0.009305
(150, 0%)	-0.014980	-0.000660	95% RaR	-0.000453	-0.000335	0.013337	-0.010955
			99% RaR	0.004739	0.005280	0.020029	-0.005395
			99.5% RaR	0.006858	0.006882	0.022337	-0.002857
			99.9% RaR	0.010181	0.011611	0.027297	0.000069
(150, 1%)	-0.018090	-0.001300	95% RaR	0.004041	0.003778	0.016434	-0.005305
			99% RaR	0.008259	0.008149	0.020423	0.001475
			99.5% RaR	0.009622	0.009174	0.021571	-0.000155
			99.9% RaR	0.012571	0.012160	0.023562	0.002789
(200, 0%)	0.017930	-0.000770	95% RaR	-0.001135	-0.001324	0.012044	-0.013086
			99% RaR	0.003721	0.003849	0.018226	-0.008325
			99.5% RaR	0.005934	0.005528	0.019928	-0.006070
			99.9% RaR	0.008855	0.010299	0.024433	-0.002158
(200, 1%)	-0.011090	-0.001090	95% RaR	0.004761	0.004620	0.022246	-0.005414
			99% RaR	0.008968	0.008931	0.025306	-0.002292
			99.5% RaR	0.010119	0.010189	0.026116	-0.001142
			99.9% RaR	0.014542	0.013384	0.028611	0.001894
MACD	-0.025460	-0.000560	95% RaR	0.008758	0.008740	-0.000057	0.008977
			99% RaR	0.012442	0.012334	0.002469	0.013314
			99.5% RaR	0.013598	0.013521	0.003376	0.015257
			99.9% RaR	0.016050	0.015370	0.005770	0.018480

All of the entries with a RaR of 0 are observed from trading strategies with long periods of moving averages (5 and 150 in the above example), and the probability distribution of the return difference is typically a sharp peak centered around 0. When longer periods of moving averages are used, the price of the latest day is less influential when making trading decisions, hence the fluctuation of price during the last 10 minutes of the latest day does not create any difference in trading decisions in the sample path from that of the perfect path.

7 Conclusion

In this paper, we quantify the achievability of backtesting results in the presence of implementation uncertainty by proposing a framework for implementing and backtesting trading strategies. We introduce the concept of RaR as a measure of the achievability of backtesting results. Moreover, we illustrate the framework on a representative class of technical trading rules, and find that a significant number of technical trading strategies with positive returns are not achievable in the presence of implementation uncertainty. Lastly, the example in Section 6.2 illustrates that our framework can be extended to trading strategies that take more than one input. In general, besides the geometric Brownian motion, we can choose another stochastic model to obtain the joint distribution of the inputs and proceed with the framework as before. More work on this extension will be carried out in the future.

In this paper, our framework assumes that market-on-close order can be used as the execution algorithm. However, strategies with large order size cause market impact, see Lo and Wang (2000). To tackle this, instead of using a market-on-close order, we may assume that a time-weighted average price (TWAP) is performed until the market closes in a linear fashion. The resulting value of RaR may then be viewed as an indicator of the robustness of the trading strategy based on its execution algorithm as well as the decision rule, and hence capturing the factor of market impact on top of the implementation uncertainty in the framework. More research on this extension will be conducted in future research projects.

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8 Appendix

8.1 Derivation of the Probability Density Function of $S^i_{1-\delta}$

Here we provide the details of deriving (5). From (1), we have

$$S^i_t = \left(O_i \exp\left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma B_t \right) \middle| S^i_0 = O_i, S^i_1 = C_i \right), \quad \text{for } t \in [0,1],$$

where B_t is the general Brownian bridge with $B_0 = 0$ and $B_1 = \frac{1}{\sigma}\left(\log\left(\frac{C_i}{O_i}\right) - \left(\mu - \frac{\sigma^2}{2}\right)\right)$. Therefore, B_t is a normally distributed random variable with mean and variance respectively given by

$$\mathbb{E}\left(B_t\right) = \frac{t}{\sigma}\left(\log\left(\frac{C_i}{O_i}\right) - \mu + \frac{\sigma^2}{2}\right),$$
$$\mathrm{Var}\left(B_t\right) = t - t^2.$$

Hence, the probability density function of B_t is

$$f_{B_t}\left(b_t\right) = \frac{1}{\sqrt{2\pi\left(t-t^2\right)}} \exp\left(- \frac{\left(b_t - \frac{t}{\sigma}\left(\log\left(\frac{C_i}{O_i}\right) - \mu + \frac{\sigma^2}{2}\right)\right)^2}{2\left(t-t^2\right)} \right), \quad \text{for } b_t \in (-\infty, \infty).$$

for $b_t \in (-\infty,\infty)$. In order to obtain the probability density function of S^i_t, we implement a change of variable $S^i_t = O_i \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right)$. Let $v : (0,\infty) \longmapsto \mathbb{R}$ be given by $v\left(s^i_t\right) = \frac{1}{\sigma}\left(\log\left(\frac{s^i_t}{O_i}\right) - \left(\mu - \frac{\sigma^2}{2}\right)t\right)$. By differentiation, we have $v'\left(s^i_t\right) = \frac{1}{\sigma s^i_t}$. Therefore,

$$\begin{aligned} f_{S^i_t}\left(s^i_t\right) &= \frac{v'\left(s^i_t\right)}{\sqrt{2\pi\left(t-t^2\right)}} \exp\left(- \frac{\left(\frac{1}{\sigma}\left(\log\left(\frac{s^i_t}{O_i}\right) - \left(\mu - \frac{\sigma^2}{2}\right)t\right) - \frac{t}{\sigma}\left(\log\left(\frac{C_i}{O_i}\right) - \mu + \frac{\sigma^2}{2}\right)\right)^2}{2\left(t-t^2\right)} \right) \\ &= \frac{1}{\sigma s^i_t \sqrt{2\pi\left(t-t^2\right)}} \exp\left(- \frac{\left(\log\left(\frac{s^i_t}{O_i}\right) - t\log\left(\frac{C_i}{O_i}\right)\right)^2}{2\sigma^2 t\left(1-t\right)} \right). \end{aligned}$$

Finally, taking $t = 1 - \delta$, we have

$$f_{S^i_{1-\delta}}\left(s^i_{1-\delta}\right) = \frac{1}{\sigma s^i_{1-\delta}\sqrt{2\pi\delta\left(1-\delta\right)}} \exp\left(-\frac{\left(\log\left(\frac{s^i_{1-\delta}}{O_i}\right) + \left(\delta - 1\right)\log\left(\frac{C_i}{O_i}\right)\right)^2}{2\sigma^2\delta\left(1-\delta\right)}\right),$$

for $s^i_{1-\delta} \in (0, \infty)$ as desired.

8.2 Derivation of the Execution Regions of VMA

The execution regions of VMA in (7) can be obtained by simple computation. Equation (7a) follows from

$$\begin{aligned}
& S^i_1 \in R_{+1} \\
\Longleftrightarrow\ & \mathrm{SMA}_{L_1,i} > \mathrm{SMA}_{L_2,i} \times \left(1 + \frac{p}{100}\right) \\
\Longleftrightarrow\ & \frac{1}{L_1} \sum_{j=i-L_1+1}^{i} S^j_1 > \frac{1}{L_2} \sum_{j=i-L_2+1}^{i} S^j_1 \times \left(1 + \frac{p}{100}\right) \\
\Longleftrightarrow\ & \frac{1}{L_1} S^i_1 + \frac{1}{L_1} \sum_{j=i-L_1+1}^{i-1} S^j_1 > \frac{1 + \frac{p}{100}}{L_2} S^1_i + \frac{1 + \frac{p}{100}}{L_2} \sum_{j=i-L_2+1}^{i-1} S^j_1 \\
\Longleftrightarrow\ & \left(\frac{1}{L_1} - \frac{1 + \frac{p}{100}}{L_2}\right) S^i_1 > \left(\frac{1 + \frac{p}{100}}{L_2} - \frac{1}{L_1}\right) \sum_{j=i-L_1+1}^{i-1} S^j_1 + \frac{1 + \frac{p}{100}}{L_2} \sum_{j=i-L_2+1}^{i-L_1} S^j_1 \\
\Longleftrightarrow\ & S^i_1 > \frac{L_1\left(1 + \frac{p}{100}\right)}{L_2 - L_1\left(1 + \frac{p}{100}\right)} \sum_{j=i-L_2+1}^{i-L_1} S^j_1 - \sum_{j=i-L_1+1}^{i-1} S^j_1 \\
\Longleftrightarrow\ & S^i_1 \in \left(M_i, \infty\right).
\end{aligned}$$

Equation (7b) follows similarly. Finally, (7c) follows from $R_0 = \mathbb{R}_{>0} \backslash \left(R_{+1} \cup R_{-1}\right) = \left[m_i, M_i\right]$.

8.3 Derivation of the Execution Regions of MACD

In the same way, the execution region of MACD in (10) can be computed as follows. Equation (10a) follows from

$$\begin{aligned}
& S^i_1 \in R_{+1} \\
\Longleftrightarrow\ & \mathrm{EMA}_{L_1,i} > \mathrm{EMA}_{L_2,i} \\
\Longleftrightarrow\ & \frac{2}{L_1+1} \sum_{k=0}^{\infty} \left(\frac{L_1 - 1}{L_1 + 1}\right)^k S^{i-k}_1 > \frac{2}{L_2+1} \sum_{k=0}^{\infty} \left(\frac{L_2 - 1}{L_2 + 1}\right)^k S^{i-k}_1 \\
\Longleftrightarrow\ & \left(\frac{2}{L_1+1} - \frac{2}{L_2+1}\right) S^i_1 > \sum_{k=1}^{\infty} \left[\frac{2}{L_2+1}\left(\frac{L_2 - 1}{L_2 + 1}\right)^k - \frac{2}{L_1+1}\left(\frac{L_1 - 1}{L_1 + 1}\right)^k\right] S^{i-k}_1 \\
\Longleftrightarrow\ & S^i_1 > \frac{\left(L_1+1\right)\left(L_2+1\right)}{2\left(L_2 - L_1\right)} \sum_{k=1}^{\infty} \left[\frac{2}{L_2+1}\left(\frac{L_2 - 1}{L_2 + 1}\right)^k - \frac{2}{L_1+1}\left(\frac{L_1 - 1}{L_1 + 1}\right)^k\right] S^{i-k}_1 \\
\Longleftrightarrow\ & S^i_1 \in \left(M_i, \infty\right).
\end{aligned}$$

Here EMA denotes exponential moving average. Equation (10b) follows a similar procedure. Finally, (10c) follows from $R_0 = \mathbb{R}_{>0} \setminus (R_{+1} \cup R_{-1}) = \{M_i\}$.