An upper bound for ex-post Sharpe ratio with application in performance measurement

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Abstract

The Sharpe ratio and the maximum drawdown (MDD) are two of the most important tools for risk measurement. Existing literatures have presented analytical results relating them under geometric Brownian motion. In this paper, we take a data-driven approach to derive a relationship between ex-post Sharpe ratio and MDD. We do not assume any specific distribution of the returns except that they be stationary and ergodic. The relationship we derive can serve as a quick sanity check for black-box performance reports if the Sharpe ratios are estimated by the ex-post Sharpe ratio. Some numerical results are given for illustration.

1 Introduction

The Sharpe ratio and the maximum drawdown are two of the most important tools to measure the risk of an investment. Since Sharpe (1966) introduces the Sharpe ratio, which is the excess return per unit of volatility, it is widely used to evaluate the performance of portfolios. On the other hand, the maximum drawdown, defined by the maximum cumulative loss from a peak to a following trough within a given time, tells investors how much they can possibly lose in the investment. The maximum drawdown is also one of the most important risk measures (de Melo Mendes and Brandi, 2004; Magdon-Ismail and Atiya, 2004).

In view of the importance of these two risk measures, previous literatures have extensively studied their analytical properties. For the Sharpe ratio, Miller and Gehr (1978) show that the ex-post Sharpe ratio is a

*Research supported by the HKRGC Grants No. CUHK300614, CUHK12500915, CUHK14306316, CUHK2/CRF/11G, AoE/M-05/12; CUHK DAG No. 4053007, and FIS Grant No. 1907303.

†The third author would like to thank Yuk Wong and Yat Wong for their help on this article as a project under The Hong Kong Academy of Gifted Education.

biased estimator for the Sharpe ratio for normally distributed returns. Jobson and Korkie (1981) derive the asymptotic distribution of the ex-post Sharpe ratio for multivariate normal returns when the length of the time series approaches infinity and find that it is an asymptotically normal estimator for the Sharpe ratio. Mertens (2002) presents the same properties for independently and identically distributed returns. Lo (2002), Christie (2005) and Opdyke (2007) further extend the results to the case of stationary and ergodic returns. For the maximum drawdown, Magdon-Ismail *et al.* (2004) derive the formula of the expected maximum drawdown under a Brownian motion with a drift, which contains integrals of series expansions. They also deduce a mathematical relationship between the expected maximum drawdown and the Sharpe ratio for the case with positive drift. Vecer *et al.* (2006) and Pospisil and Vecer (2008) employ partial differential equation approaches to compute the distribution and the expected value of the maximum drawdown respectively under geometric Brownian motions. Zhang and Hadjiliadis (2012), Zhang *et al.* (2013) and Zhang (2015) study drawdowns under different processes from probabilistic perspectives, they then apply the results in valuation of drawdown insurances. Carr *et al.* (2011) introduce vanilla drawdown insurance contracts and semi-static replication strategies based on options. Landriault *et al.* (2015) extend the study from the magnitude of drawdowns to the frequency rate of drawdowns, they also propose insurances protecting against the risk of frequent drawdowns. For the time of drawdowns, Taylor (1975) studies the first passage time of drawdowns exceed a threshold and the running maximum at the first passage time under Brownian motions. Later, Lehoczky *et al.* (1977) generalize it to time homogeneous diffusion processes. Zhang and Hadjiliadis (2012) study the drawdown time and the speed of market crash under general diffusion processes. Mijatović and Pistorius (2012) investigate the first passage time and random variables associated with it under spectrally negative Lévy processes. Landriault *et al.* (2017) examine the asymptotics of the magnitude of drawdowns when the threshold approaches zero, they also investigate the duration of drawdowns under Lévy processes with two-sided jumps. On the other hand, there are some studies on the probabilistic behavior of both drawdown and drawup processes. Hadjiliadis and Večeř (2006), Pospisil *et al.* (2009), Zhang and Hadjiliadis (2010) and Zhang (2015) derive the probability that a rally precedes a drawdown in different models. Salminen and Vallois (2007) compute the joint distribution of the maximum drawdown and the maximum drawup for a Brownian motion up to an independent exponential time.

While most of the articles consider the analytical properties only under a specific distribution, here in this paper, we consider the mathematical relationship of the ex-post Sharpe ratio and the maximum drawdown by using the data from reported profits and losses. We do not assume any specific distribution of the returns except that they be stationary and ergodic. We will show that the ex-post Sharpe ratio and the maximum drawdown are mutually bounded. Our results can serve as a quick sanity check for black-box performance reports in practice. For instance, suppose one receives an annual report indicating that the value of the portfolio was initially 1 billion but ends up with 1.8 billion in a year, yielding an ex-post Sharpe ratio and a maximum drawdown of 1.2 and 10% respectively. Is this report valid? At first glance, the report does not show any abnormality. However, at the end of this paper we will apply our results to show that it cannot be the case.

This paper is organized as follows: In Section 2, we give the derivations of the bounds. In Section 3, we show some numerical results to illustrate the bounds for the risk measures. Section 4 gives an example of how to apply the bounds to check the validity of given data. Section 5 gives the conclusion.

2 Bounds for the risk measures

2.1 Definitions and notations

Given a time series data of market-to-market values at discrete time $\{a_0, a_1, a_2, ..., a_n\}$, we evaluate its performance over the period using two definitions of return: logarithmic return and holding period return.

Logarithmic return is one of the most popular definitions of return and it is defined by $d_{1,i} = \ln(\dfrac{a_{i+1}}{a_i})$ at time i, for $i = 0, 1, ..., n-1$. Financial asset price series are assumed to be with lognormal distribution in various mathematical models. Under such assumption, the logarithmic returns are normally distributed.

However, if we consider a buy-and-hold strategy for profit and loss induced by market-to-market mechanism, the most reasonable way to evaluate this strategy is to use holding period return $\dfrac{a_n - a_0}{a_0}$ (see Christopherson *et al.*, 2009, Chapter 3). The subperiod return of the strategy between time i and time $i+1$ is given by $d_{2,i} = \frac{(a_{i+1}-a_i)}{a_0}$ for $i = 0, 1, ..., n-1$. We will call this the subperiod holding period return, SHP return in short. This is just the actual payoff at time i normalized by the initial capital a_0.

These two definitions of return are the mainstream methods for performance evaluation and they render the arithmetic sum of the return more relevant to the average return over the whole period.

It has been shown that the ex-post Sharpe ratio S is an asymptotically normal estimator of the Sharpe ratio R (S and R are defined mathematically below) under milder assumptions on the returns when time advances (Miller and Gehr, 1978; Jobson and Korkie, 1981; Mertens, 2002; Lo, 2002; Christie, 2005; Opdyke, 2007). The asymptotic normality of the ex-post Sharpe ratio has been shown by Opdyke (2007) under the assumption of stationary and ergodic returns. In this paper we are going to consider the relationship between the ex-post Sharpe ratio and other measures; we therefore assume the returns are stationary and ergodic throughout this paper.

For the return $d_{1,i}$ (logarithmic return) and $d_{2,i}$ (SHP return) defined above, the Shape ratios R_j are defined by:

$$R_j = \frac{\mathbb{E}(d_{j,i}) - r}{\sqrt{\mathrm{Var}(d_{j,i})}}, \quad j = 1, 2,$$

where r denotes the risk-free rate per unit of time, $\mathbb{E}(d_{j,i})$ and $\mathrm{Var}(d_{j,i})$ denote the expected value and variance of returns $d_{j,i}$. Also the ex-post Sharpe ratios S_j are defined by:[1]

$$S_j = \frac{\bar{d}_j - r}{\sigma_j}, \quad j = 1, 2, \tag{1}$$

where $\bar{d}_j = \frac{1}{n}\sum_{i=0}^{n-1} d_{j,i}$ and $\sigma_j = [\frac{1}{n}\sum_{i=0}^{n-1}(d_{j,i} - \bar{d}_j)^2]^{1/2}$. For details about ex-ante and ex-post Sharpe ratio, see Sharpe (1966) and Sharpe (1994).

The maximum drawdown M is defined as the maximum cumulative loss normalized by the peak value before the loss (Chekhlov *et al.*, 2005):

$$M = \max_{0 \le t \le \tau \le n} \frac{(a_t - a_\tau)}{a_t}. \tag{2}$$

2.2 Main Results

Using the definitions of logarithmic return, the induced ex-post Sharpe ratio and the maximum drawdown, we derive the following three inequalities bounding the ex-post Sharpe ratio S_1, the maximum drawdown M and the average return $\bar{d}_1$ respectively:

(a) an upper bound for the maximum drawdown if $M \neq 1$ and $S_1 \neq 0$:

$$M \le 1 - e^{-\frac{n}{2}\left(\sqrt{(\frac{\bar{d}_1 - r}{S_1})^2 + \bar{d}_1^2} - \bar{d}_1\right)}, \tag{3}$$

(b) an upper bound for the ex-post Sharpe ratio if $M \neq 1$ and $S_1 \neq 0$:

$$S_1 \le \frac{n(\bar{d}_1 - r)}{2}\sqrt{\frac{1}{\ln(1-M)[\ln(1-M) - n\bar{d}_1]}}, \tag{4}$$

(c) a lower bound for the average return if $M \neq 1$, $S_1 \neq 0$, $\bar{d}_1 > 0$ and $(\ln(1-M)S_1)^2 \ge \frac{n(n-1)r^2}{4}$:

$$\bar{d}_1 \ge \frac{nr - 2\ln(1-M){S_1}^2 + 2|S_1|\sqrt{[\ln(1-M)]^2({S_1}^2 + 1) - nr\ln(1-M)}}{n}. \tag{5}$$

For the case of SHP return, one can derive similarly three inequalities bounding the induced ex-post Sharpe ratio S_2, the maximum drawdown M, and the average return $\bar{d}_2$ respectively:

(a) an upper bound for the maximum drawdown if $S_2 \neq 0$:

$$M \leq \frac{n}{2}\left(\sqrt{\left(\frac{\bar{d}_2 - r}{S_2}\right)^2 + \bar{d}_2^2}\right) - \bar{d}_2), \tag{6}$$

(b) an upper bound for the ex-post Sharpe ratio if $S_2 \neq 0$:

$$S_2 \leq \frac{n(\bar{d}_2 - r)}{2}\sqrt{\frac{1}{M(M + n\bar{d}_2)}}, \tag{7}$$

(c) a lower bound for the average return if $S_2 \neq 0$, $\bar{d}_2 > 0$ and $(MS_2)^2 \geq \frac{n^2r^2}{4}$:

$$\bar{d}_2 \geq \frac{nr + 2M{S_2}^2 + 2|S_2|\sqrt{M^2({S_2}^2 + 1) + nrM}}{n}. \tag{8}$$

Hence, once the length of the time series n, the risk-free rate r and the average return $\bar{d}_j$ are known, we are able to find an upper bound of the ex-post Sharpe ratio when the maximum drawdown is given and vice versa. Moreover, when the ex-post Sharpe ratio and the maximum drawdown are given, we are able to find a lower bound of the average return. The upper bounds for the ex-post Sharpe ratios (4) and (7) are the key results in this paper. Most of the existing studies have been focused on the probabilistic behaviors of risk measures under specific distributions (see, *e.g.*, Magdon-Ismail and Atiya (2004) and Pospisil and Vecer (2008)). Our bounds on the risk measures, however, are derived based on empirical data and do not assume any specific distribution of the returns except that they be stationary and ergodic. Moreover, the assumptions of the performance measures are mild and can usually be satisfied in practice (see *Remark* below). Performance reports stating Sharpe ratios and maximum drawdowns may be subject to miscalculation or even fraud to lure potential investors. Our bounds can serve as a sanity check on the consistency among the performance measures shown in a report. If fact, our bounds can be applied as long as the definition of the ex-post Sharpe ratio (1) is adopted to estimate the Sharpe ratio.

Remark: In practice, $M \neq 1$ can usually be satisfied because it is rare for a portfolio value to drop to zero. Moreover the average return of a portfolio is seldom exactly equal to the risk free rate, hence $S_1 \neq 0$ and $S_2 \neq 0$ in most situations. This is also the case for $(\ln(1 - M)S_1)^2 \geq \frac{n^2r^2}{4}$ and $(MS_2)^2 \geq \frac{n^2r^2}{4}$ because usually the risk-free rate r is small. Moreover, if one adopts the risk-free rate $r = 0$ in the definition of the ex-post Sharpe ratio (see, *e.g.*, Keating and Shadwick (2002), Magdon-Ismail *et al.* (2004)), then $(\ln(1 - M)S_1)^2 \geq \frac{n^2r^2}{4}$ and $(MS_2)^2 \geq \frac{n^2r^2}{4}$ hold automatically.

2.3 Derivations

In this section we only illustrate the derivations of the bounds for the case of logarithmic return, however the derivations of the bounds for the case of SHP return are similar.

Let the maximum drawdown starts at time k and it lasts for p time steps, i.e. it ends at time $k+p$. Moreover assume that the maximum drawdown $M \neq 1$, i.e. the value of the time series data never drops to 0. Let $\hat{M} = \ln(a_k/a_{k+p})$, it can be written as:

$$\hat{M} = -\sum_{i=k}^{k+p-1} d_{1,i}.$$

Applying the Cauchy-Schwarz inequality (Hardy *et al.*, 1952, Chapter 2), we obtain:

$$\sum_{i=k}^{k+p-1} d_{1,i}^2 \geq \frac{(\sum\limits_{i=k}^{k+p-1} d_{1,i})^2}{p} = \frac{\hat{M}^2}{p}.$$

Applying the Cauchy-Schwarz inequality again on $(\sum_{i=0}^{k-1} d_{1,i} + \sum_{i=k+p}^{n-1} d_{1,i})$, we obtain:

$$\sum_{i=0}^{k-1} d_{1,i}^2 + \sum_{i=k+p}^{n-1} d_{1,i}^2 \geq \frac{(\sum\limits_{i=0}^{k-1} d_{1,i} + \sum\limits_{i=k+p}^{n-1} d_{1,i})^2}{n-p} = \frac{(n\bar{d}_1 + \hat{M})^2}{n-p}.$$

Summing up the above two inequalities, we have:

$$\sum_{i=0}^{n-1} d_{1,i}^2 \geq \frac{\hat{M}^2}{p} + \frac{(n\bar{d}_1 + \hat{M})^2}{n-p}. \tag{9}$$

For $p \in [0, n]$, the R.H.S. of (9) attains its minimum if $-\frac{(\hat{M})^2}{p^2} + \frac{(n\bar{d}_1+\hat{M})^2}{(n-p)^2} = 0$, i.e. $p = \frac{n\hat{M}}{n\bar{d}_1+2\hat{M}}$. Hence by substituting $p = \frac{n\hat{M}}{n\bar{d}_1+2\hat{M}}$ in the R.H.S. of (9), we have:

$$\sum_{i=0}^{n-1} d_{1,i}^2 \geq \frac{(n\bar{d}_1 + 2\hat{M})^2}{n}.$$

Assuming the ex-post Sharpe ratio $S_1 \neq 0$ and after some simplifications and rearrangements, we have:

$$\hat{M} \leq \frac{n}{2}\left(\sqrt{(\frac{\bar{d}_1 - r}{S_1})^2 + \bar{d}_1^2} - \bar{d}_1\right).$$

By the fact that $M = 1 - e^{-\hat{M}}$, we obtain the upper bound for the maximum drawdown:

$$M \leq 1 - e^{-\frac{n}{2}\left(\sqrt{(\frac{\bar{d}_1 - r}{S_1})^2 + \bar{d}_1^2} - \bar{d}_1\right)}. \tag{10}$$

Assume that the maximum drawdown $M \neq 1$ and the ex-post Sharpe ratio $S_1 \neq 0$. Then by (10), we obtain the upper bound for the ex-post Sharpe ratio:

$$S_1 \leq \frac{n(\bar{d}_1 - r)}{2} \sqrt{\frac{1}{\ln(1-M)[\ln(1-M) - n\bar{d}_1]}}. \tag{11}$$

On the other hand, assuming $M \neq 1$, $S_1 \neq 0$, $\bar{d}_1 > 0$ and $(\ln(1-M)S_1)^2 \geq \frac{n^2 r^2}{4}$, and after some rearrangement of (10), we get:

$$n^2 \bar{d}_1^2 - [2n^2 r - 4n \ln(1-M){S_1}^2]\bar{d}_1 + n^2 r^2 - 4[\ln(1-M)]^2 {S_1}^2 \geq 0. \tag{12}$$

The L.H.S. of (12) is a quadratic expression of $\bar{d}_1$. Its two roots are

$$\bar{d}_{1,\pm} = \frac{nr - 2\ln(1-M){S_1}^2 \pm 2|S_1|\sqrt{[\ln(1-M)]^2({S_1}^2 + 1) - nr\ln(1-M)}}{n}.$$

Thus the solution of (12) is given by $\bar{d}_1 \leq \bar{d}_{1,-}$ or $\bar{d}_1 \geq \bar{d}_{1,+}$. Since $(\ln(1-M)S_1)^2 \geq \frac{n^2 r^2}{4}$, we can easily show that $\bar{d}_{1,-} \leq 0$. Hence under the assumption that $\bar{d}_1 > 0$, we obtain the lower bound for the average return:

$$\bar{d}_1 \geq \frac{nr - 2\ln(1-M){S_1}^2 + 2|S_1|\sqrt{[\ln(1-M)]^2({S_1}^2 + 1) - nr\ln(1-M)}}{n}. \tag{13}$$

As discussed in Section 2.2, the assumptions we make through out the derivations of the bounds are mild which can usually be satisfied in real situation. Thus our results have practical value and can serve as a sanity check for performance reports.

3 Numerical results

In this section, we give some numerical results to illustrate the bounds derived in Section 2. In Section 3.1, we illustrate with a set of given parameters. In Section 3.2, we illustrate with a set of historical data of the Standard & Poor's 500 constituent stocks.

3.1 Illustration with given parameters

We use a set of given parameters to illustrate the behaviors of the bounds. In particular, we choose risk-free rate $r = 2 \times 10^{-5}$; average return $\bar{d}_1$ or $\bar{d}_2 = 0.006, 0.012, 0.024$ and length of period $n = 250, 500, 1000$ and show how the bounds behave when the parameters are changing.

First, we focus on the case of logarithmic return. Figures 1 and 2 illustrate the upper bound of the maximum drawdown M for various magnitudes of S_1 and n and for various magnitudes of S_1 and $\bar{d}_1$ respectively, see (3). Figures 3 and 4 illustrate the upper bound of the ex-post Sharpe ratio S_1 for various magnitudes of M and n and for various magnitudes of M and $\bar{d}_1$ respectively, see (4).

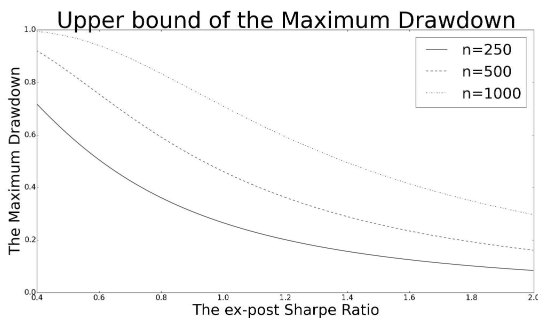

Figure 1: Upper bound of M for $\bar{d}_1 = 0.006$ and different n

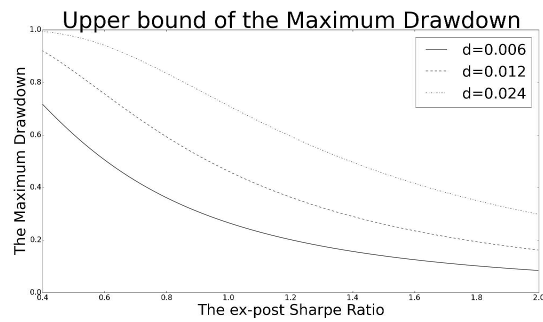

Figure 2: Upper bound of M for $n = 250$ and different $\bar{d}_1$

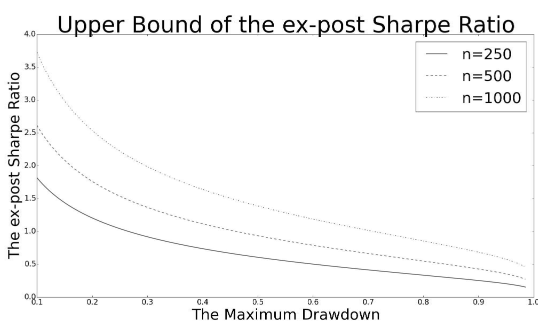

Figure 3: Upper bound of S_1 for $\bar{d}_1 = 0.006$ and different n

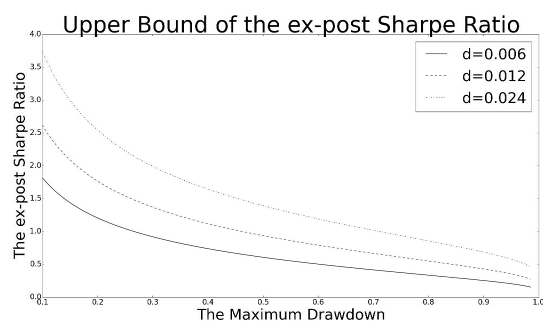

Figure 4: Upper bound of S_1 for $n = 250$ and different $\bar{d}_1$

In the figures, the region below the curve is the feasible region for the risk measures and it is impossible for the risk measures to lie above the curve. One can observe that we have a tighter upper bound when the magnitude of the given risk measure is greater, and we have a looser upper bound when the length of the period is longer as more freedom is given to the time series data. Moreover, doubling the length of the period n and doubling the average return $\bar{d}_1$ give almost the same effect on the upper bounds.

Table 1 illustrates the lower bound of the average return $\bar{d}_1$ for different magnitudes of S_1 and M when the length of the period $n = 250$, see (5).

$S_1 \backslash M$	0.1	0.3	0.5	0.7	0.9
0.4	0.0005	0.0017	0.0033	0.0057	0.0109
0.8	0.0014	0.0048	0.0093	0.0161	0.0307
1.2	0.0028	0.0095	0.0184	0.0320	0.0611
1.6	0.0047	0.0160	0.0310	0.0538	0.1028
2.0	0.0072	0.0242	0.0470	0.0816	0.1561

Table 1: Lower bound of the average return $\bar{d}_1$ for different S_1 and M

Under the assumption that $M \neq 1$, $S_1 \neq 0$, $\bar{d}_1 > 0$ and $(\ln(1-M)S_1)^2 \geq \frac{n^2 r^2}{4}$, if n, r, S_1 and M are known, we can always find the lower bound of the average return $\bar{d}_1$. If one fixes the ex-post Sharpe ratio S_1 and increase the maximum drawdown M, the lower bound of the average return $\bar{d}_1$ will be greater. The reason is that the volatility will be higher with a higher maximum drawdown, which forces the average return to be higher in order to keep the ex-post Sharpe ratio S_1 unchanged. On the other hand, if one fixes the maximum drawdown M and raise the ex-post Sharpe ratio S_1, the lower bound of the average return $\bar{d}_1$ will also be higher. Since the average return $\bar{d}_1$ needs to be greater so as to yield a higher ex-post Sharpe ratio S_1.

Next, we show the behaviors of the bounds for the case of SHP return. Figures 5 and 6 illustrate the upper bound of the maximum drawdown M for various magnitudes of S_2 and n and for various magnitudes of S_2 and $\bar{d}_2$ respectively, see (6). Figures 7 and 8 illustrate the upper bound of the ex-post Sharpe ratio S_2 for various magnitudes of M and n and for various magnitudes of M and $\bar{d}_2$ respectively, see (7). Table 2 illustrates the lower bound of the average return $\bar{d}_2$ for different magnitudes of S_2 and M when the length of the period $n = 250$, see (8). We see from the figures and table that the behaviors of the bounds for the case of SHP return are similar to the case of logarithmic return.

$S_2 \backslash M$	0.1	0.3	0.5	0.7	0.9
0.4	0.0005	0.0014	0.0024	0.0033	0.0043
0.8	0.0014	0.0040	0.0067	0.0094	0.0120
1.2	0.0027	0.0080	0.0133	0.0186	0.0239
1.6	0.0045	0.0134	0.0224	0.0313	0.0402
2.0	0.0068	0.0204	0.0339	0.0475	0.0610

Table 2: Lower bound of the average return $\bar{d}_2$ for different S_2 and M

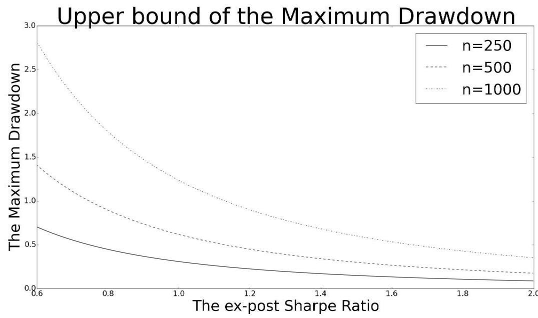

Figure 5: Upper bound of M for $\bar{d}_2 = 0.006$ and different n

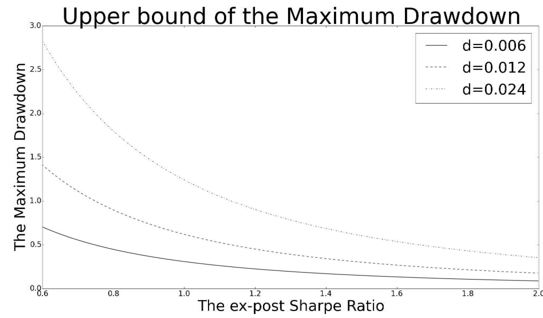

Figure 6: Upper bound of M for $n = 250$ and different $\bar{d}_2$

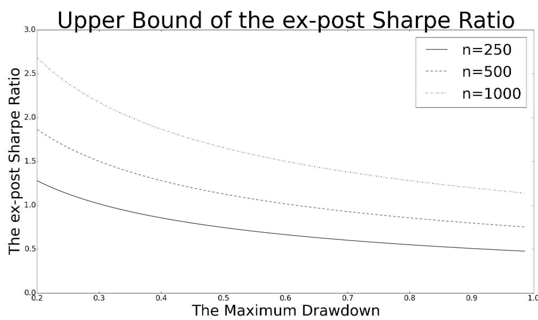

Figure 7: Upper bound of S_2 for $\bar{d}_2 = 0.006$ and different n

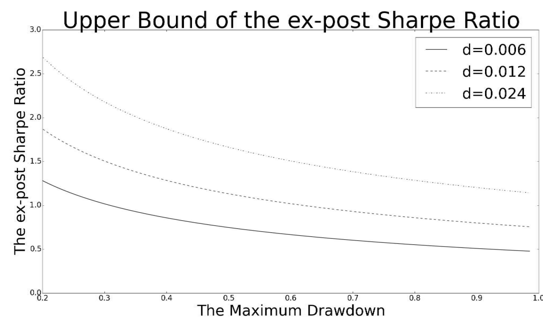

Figure 8: Upper bound of S_2 for $n = 250$ and different $\bar{d}_2$

3.2 Illustration with historical data

We first collect all of the Standard & Poor's 500 constituent stocks' daily closing data over the period of January 1, 2015 through December 31, 2015. Then we choose two groups of stocks with average return $0.0005 \leq \bar{d}_1 \leq 0.001$ and $0.001 \leq \bar{d}_1 \leq 0.002$ respectively. There are 86 stocks in the former group and 42 in the latter.

In Figures 9 and 10, the red dots represent the magnitude of the ex-post Sharpe ratio and the maximum drawdown of the stocks with return $0.0005 \leq \bar{d}_1 \leq 0.001$, while the curves represent the upper bound for the ex-post Sharpe ratio and the maximum drawdown respectively using a rounded down average return $\bar{d}_1 = 0.0005$. Figures 11 and 12 are similar but for the stocks with average return $0.001 \leq \bar{d}_1 \leq 0.002$ and we plot the curves using a rounded down average return $\bar{d}_1 = 0.001$.

In Figures 9 to 12, all of the risk measures of the constituent stocks lie below the curve. This shows the validity of the upper bounds we derived for the ex-post Sharpe ratio and the maximum drawdown.

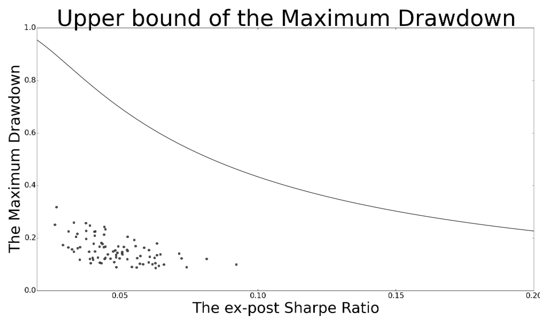

Figure 9: Upper bound of M for the S&P 500 constituents with $0.0005 \leq \bar{d}_1 \leq 0.001$

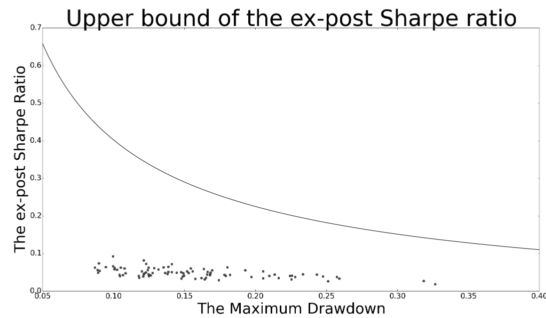

Figure 10: Upper bound of S_1 for the S&P 500 constituents with $0.0005 \leq \bar{d}_1 \leq 0.001$

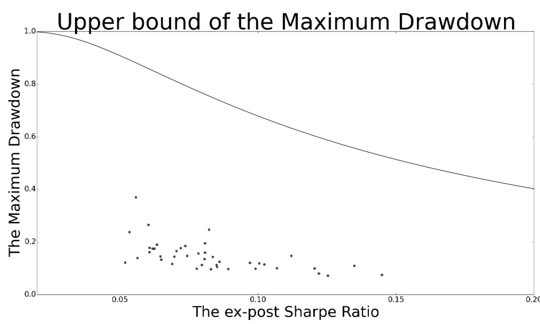

Figure 11: Upper bound of M for the S&P 500 constituents with $0.001 \leq \bar{d}_1 \leq 0.002$

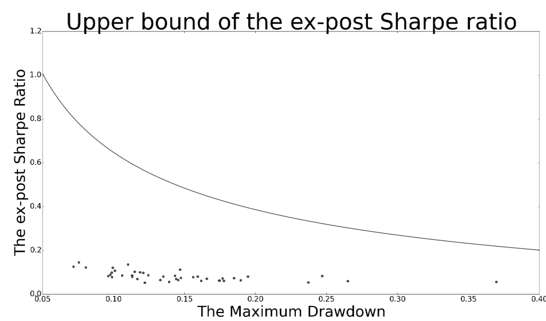

Figure 12: Upper bound of S_1 for the S&P 500 constituents with $0.001 \leq \bar{d}_1 \leq 0.002$

4 Discussion

In the last section, we see that data from the market satisfy our theoretical bounds. Conversely, using our bounds, we can determine whether data reported in any financial statement are valid or not. As an example, consider the data of the portfolio we mentioned in Section 1, i.e. a portfolio value grows from 1 billion to 1.8 billion in a year and generates an ex-post Sharpe ratio and a maximum drawdown of 1.2 and 10% respectively. With 250 trading days a year, we have $n = 249$, $M = 0.1$ and $\bar{d}_1 = \frac{\ln(1.8)}{249} = 0.0024$. Thus, by (4), the portfolio cannot have a ex-post Sharpe ratio greater than 1.09 even with zero risk-free rate. Hence, the claim with ex-post Sharpe ratio of 1.2 must be false.

5 Conclusion

In this paper, we adopt a data-driven approach to derive the relationship between ex-post Sharpe ratio and maximum drawdown. For both definitions of return (logarithmic return and holding period return) our bounds do not assume any specific distribution of the returns but assume it to be stationary and ergodic. The results can serve as a quick sanity check for black-box performance reports if the definition of ex-post Sharpe ratio (1) is used to estimate the Sharpe ratios. Our numerical results illustrate how ex-post Sharpe ratio and maximum drawdown are mutually bounded.

Notes

[1]Sharpe (1994) states that the population standard deviation can be used in the definition of the ex-post Sharpe ratio when the lengths of periods n are the same for the portfolios being compared. This is because the relative magnitudes of the ex-post Sharpe ratios would be the same.

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