MATH1010A-H Assignment 2 Due date 23-10-2015

(1) Find, from first principles, the first derivative of the functions below: (5 points each)

(a) $f(x) = \dfrac{x^2}{x+1}$.

(b) $g(x) = x^{2/3}$.

(2) Compute the limits below. (2 points each) (Hint: Use continuity.)

(a) $\lim_{x\to 0} e^{\sqrt{|\sin x|}}$.

(b) $\lim_{x\to \pi} \ln(1 + |\cos x|)$.

(3) Let $f(x) = \cos x - 1 + \dfrac{x^2}{2}$, show that $f(x)$ is an increasing function on $[0, +\infty)$ and hence show that $\cos x \geq 1 - \dfrac{x^2}{2}$. (6 points)

(4) If $f(2) = 2$, $f'(2) = 3$, $g(2) = 4$, $g'(2) = 5$, compute

(a) $\dfrac{d}{dx}(f(x)g(x))\Big|_{x=2}$.

(b) $\dfrac{d}{dx}(\dfrac{f(x)}{g(x)})\Big|_{x=2}$.

(c) $\dfrac{d}{dx}(g(f(x)))\Big|_{x=2}$.

(2 points each)

(5) Let A be a constant. Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by

$$f(x) = \begin{cases} x + A & \text{if } x \geq 1, \\ x^2 - x + 1 & \text{if } x < 1. \end{cases}$$

Suppose $f(x)$ is a continuous function on $\mathbf{R}$.

(a) Find the value of A. (3 points)

(b) Show that f is differentiable on $\mathbf{R}$. (3 points)

(6) Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

(a) Show that f is a continuous function on $\mathbf{R}$. (4 points)

(b) (i) Show that f is differentiable at 0. (4 points)

(ii) Find $f'(x)$ explicitly for all $x \in \mathbf{R}$. (2 points)

(c) Is f' differentiable at 0? Justify your answer. (4 points)

(7) Let $n = 0, 1, 2$. Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by

$$f(x) = \begin{cases} x^n \cos^2 \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Suppose $n = 0$. Is f continuous at $x = 0$? Justify your answer. (4 points)

(b) Suppose $n = 1$.

(i) Is f continuous at $x = 0$? Justify your answer. (4 points)

(ii) If f differentiable on $\mathbf{R}$? Justify your answer. (4 points)

(iii) Find an explicit formula of f'. (2 points)

(iv) Is f' continuous on $\mathbf{R}$? Justify your answer. (3 points)

(c) Suppose $n = 2$.

(i) Is f continuous at $x = 0$? Justify your answer. (4 points)

(ii) If f differentiable on $\mathbf{R}$? Justify your answer. (4 points)

(iii) Find an explicit formula of f'. (2 points)

(iv) Is f' continuous on $\mathbf{R}$? Justify your answer. (3 points)

(8) Suppose $f, g : \mathbf{R} \to \mathbf{R}$. Determine if each of the following statement is true or false. If true, give reasons. If false, provide a counter example. (4 points each)

(a) If $f(x)$ is continuous at $x = c$ but $g(x)$ is **not** continuous at $x = c$, then $f(x) + g(x)$ is **not** continuous at $x = c$.

(b) If both $f(x)$ and $g(x)$ are **not** continuous at $x = c$, then $f(x) + g(x)$ is **not** continuous at $x = c$.

(c) If $f(x)$ is continuous at $x = c$ but $g(x)$ is **not** continuous at $x = c$, then $f(x)g(x)$ is **not** continuous at $x = c$.

(d) If $f(x)$ is continuous at $x = c$ but $g(x)$ is **not** continuous at $x = f(c)$, then $g(f(x))$ is **not** continuous at $x = c$.

(9) Recall the following result:

- **Theorem.** Let $f, g : \mathbf{R} \to \mathbf{R}$ be continuous functions. Then $f(g(x))$ is also a continuous function on $\mathbf{R}$.

Suppose $g(x)$ is a continuous function on $\mathbf{R}$. Using the theorem above or otherwise, show that:

(a) The function $\sqrt{|g(x)|}$ is a continuous function on $\mathbf{R}$. (4 points)

(b) The function

$$h(x) = \begin{cases} g(x) & \text{if } g(x) \geq 0, \\ 0 & \text{if } g(x) < 0. \end{cases}$$

is a continuous function on $\mathbf{R}$. (4 points)