MATH1010A-H Assignment 1 New due date 9-10-2015

1. Give the general shape of the graph of each of the functions below on $\mathbb{R}$. (4 points each)
There is no need to use calculus. Actually you may get into a mess if you blindly use calculus in some parts.

(a) $f(x) = |1 + 3\sin 2x|$
Hint: First draw the graph of the function $1 + 3\sin 2x$.

(b) $f(x) = 3\sin x + 4\cos x$
Hint: Can you express $3\sin x + 4\cos x$ in the form $A\sin(x+\theta)$, where A, θ are constants?

(c) $f(x) = x^3 \cos x$
Hint: The pair of curves $y = x^3$, $y = -x^3$ split the plane into four regions. Which regions do you expect the graph of this function to pass through?

(d) $f(x) = |3 - |x^2 - 1||$
Hint: Re-write the 'formula of definition' for this function as that for a piecewise defined function. Alternatively, begin with the curve $y = x^2 - 1$ and ask what the curve $y = |x^2 - 1|$ looks like.

(e) $f(x) = \begin{cases} 5 - \ln(x^2 + 4x + 4) & \text{if} \quad x \neq -2 \\ 3 & \text{if} \quad x = -2 \end{cases}$
Hint: Simplify $\ln(x^2 + 4x + 4)$ first.

(f) $f(x) = \begin{cases} 1 - |x + 3| & \text{if} \quad x < -2 \\ x|x| & \text{if} \quad -2 \le x < 2 \\ e^{2(1+\ln 2)-x} & \text{if} \quad x \ge 2 \end{cases}$

2. Consider each of the limits below. Determine whether it exists or not. If it does, compute its value as well. If it does not, explain why not. (4 points each)
There is no need to apply L'Hôpital's Rule.

(a) $\displaystyle\lim_{t\to 0} \frac{\ln(e+t) - 1}{et}$

(b) $\displaystyle\lim_{x\to -2} \frac{|x|^3 - 8}{x^4 - 16}$
Remark. What actually is $\dfrac{|x|^3 - 8}{x^4 - 16}$ when x is slightly greater than -2? Or slightly less than -2?

(c) $\displaystyle\lim_{x\to 1} \frac{\sqrt{x^2 - 2x + 1}}{x - 1}$
Remark. Beware of the square root!

(d) $\displaystyle\lim_{x\to \pi} \frac{|\sin x|}{e^x - e^\pi}$

(e) $\displaystyle\lim_{x\to 0} \frac{x}{|x|} \sin \frac{|x|}{x}$

(f) $\displaystyle\lim_{x\to 0} \frac{\sin^2 x}{1 - \cos x}$

(g) $\displaystyle\lim_{x\to +\infty} \frac{8e^{2x} - 9x^2}{(6e^x + x^4)^2}$

(h) $\displaystyle\lim_{x\to \infty} \frac{\sin x}{x}$

(i) $\displaystyle\lim_{x\to -\infty} \left(x + \sqrt{x^2 + 4x}\right)$

(j) $\displaystyle\lim_{x\to +\infty} \frac{\cos x}{(\ln x)^2 + 2015}$

(k) $\displaystyle\lim_{x\to -\infty} \left(\sqrt{x^2 + x + 1} - \sqrt{x^2 - x + 1}\right)$

3. Let $\beta \in (1, +\infty)$. Let $f : (0, +\infty) \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^\beta + \beta - 1 - \beta x$ for any $x \in (0, +\infty)$.
   (a) (i) Compute f'. (2 points)
       (ii) Show that f is strictly decreasing on $(0, 1]$. (1 point)
       (iii) Show that f is strictly increasing on $[1, +\infty)$. (1 point)
       (iv) Determine whether f attains the maximum and/or the minimum on $(0, +\infty)$. (2 points)
   (b) Hence, or otherwise, show that $(1 + r)^\beta \geq 1 + \beta r$ for any $r \in (-1, +\infty)$. (2 points)

4. Let $f : \mathbb{R}\backslash\{-1, 1\} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{3x - 5}{x^2 - 1}$ for any $x \in \mathbb{R}$. Consider the curve $C : y = f(x)$.
   (a) Find the x- and y-intercepts of the curve C. (2 points)
   (b) Find f', f''. (3 points)
   (c) Find the (relative) maximum and (relative) minimum points of the curve C. (3 points)
   (d) What are the asymptotes of the curve C, if any? (2 points)
   (e) Sketch the curve C for $-6 \leq x \leq 6$. (4 points)

5. Water is being drained, at a constant rate of 1/4 cubic meter per hour, from the bottom of a container that takes the shape of an inverted regular cone. The radius of the base of the cone is 3 meters, and the height of the cone is 2 meters. What is the rate of change of the depth of the water, when the water is 1 meter deep? (10 points)