Math 2230B, Complex Variables with Applications

1. By differentiating the Maclaurin series representation

$$\frac{1}{1-z} = \Sigma_{n=0}^{\infty} z^n \quad (|z| < 1)$$

obtain the expansions

$$\frac{1}{(1-z)^2} = \Sigma_{n=0}^{\infty}(n+1)z^n \quad (|z| < 1)$$

and

$$\frac{1}{(1-z)^3} = \Sigma_{n=0}^{\infty}(n+1)(n+2)z^n \quad (|z| < 1)$$

2. By substituting $\frac{1}{1-z}$ for z in the expression

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty}(n+1)z^n \quad (|z| < 1),$$

found in Exercise 1, derive the Laurent series representation

$$\frac{1}{z^2} = \sum_{n=2}^{\infty} \frac{(-1)^n(n-1)}{(z-1)^n} \quad (1 < |z-1| < \infty).$$

(Compare with Example 2, Sec. 71.)

3. Find the Taylor series for the function

$$\frac{1}{z} = \frac{1}{2+(z-2)} = \frac{1}{2} \cdot \frac{1}{1+(z-2)/2}$$

about the point $z_0 = 2$. Then, by differentiating that series term by term, show that

$$\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n \quad (|z-2| < 2).$$

4. Show that the function defined by means of the equations

$$f(z) = \begin{cases} (1-\cos z)/z^2 & \text{when } z \neq 0, \\ 1/2 & \text{when } z = 0 \end{cases}$$

is entire.(See example 1, Sec. 71.)

5. Prove that if

$$f(z) = \begin{cases} \frac{\cos z}{z^2-(\pi/2)^2} & \text{when } z \neq \pm\pi/2, \\ -\frac{1}{\pi} & \text{when } z = \pm\pi/2, \end{cases}$$

then f is an entire function.

6. In the ω plane, integrate the Taylor series expansion (see Example 1, Sec. 64)

$$\frac{1}{w} = \sum_{n=0}^{\infty}(-1)^n(w-1)^n \quad (|w-1|<1)$$

along a contour interior to its circle of convergence from $\omega = 1$ to $\omega = z$ to obtain the representation

$$Logz = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n \quad (|z-1|<1).$$

7. Use the result in Exercise 6 to show that if

$$f(z) = \frac{Logz}{z-1} \quad \text{when } z \neq 1$$

and f(1)=1, then f is analytic throughout the domain

$$0 < |z| < \infty, -\pi < \operatorname{Arg} z < \pi.$$

8. Prove that if f is analytic at z_0 and $f(z_0) = f'(z_0) = ... = f^{(m)}(z_0) = 0$, then the function g defined by means of the equations

$$g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^{m+1}} & \text{when } z \neq z_0 \\ \frac{f^{(m+1)}(z_0)}{(m+1)!} & \text{when } z = z_0 \end{cases}$$

is analytic at z_0.

9. Consider two series

$$S_1(z) = \sum_{n=0}^{\infty} a_n\left(z-z_0\right)^n, \quad S_2(z) = \sum_{n=1}^{\infty}\frac{b_n}{\left(z-z_0\right)^n},$$

which converge in some annular domain centered at z_0. Let C denote any contour lying in that annulus, and let g(z) be a function which is continuous on C. Modify the proof of Theorem 1, Sec. 71, which tells us that

$$\int_C g(z)S_1(z)dz = \sum_{n=0}^{\infty} a_n \int_C g(z)\left(z-z_0\right)^n dz,$$

to prove

$$\int_C g(z) S_2(z) dz = \sum_{n=1}^{\infty} b_n \int_C \frac{g(z)}{(z-z_0)^n} dz.$$

Conclude from these results that if

$$S(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n},$$

then

$$\int_C g(z) S(z) dz = \sum_{n=-\infty}^{\infty} c_n \int_C g(z) (z-z_0)^n dz.$$