THE CHINESE UNIVERSITY OF HONG KONG
MATH2230 Tutorial 4

(Prepared by Tai Ho Man)

Definition 1. *Let $w(t) = u(t) + iv(t)$ be a complex function of a real variable t, the definite integral of $w(t)$ over the interval $a \leq t \leq b$ is defined as*

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

Definition 2. *Let $z(t) = x(t) + iy(t) : [a, b] \to \mathbb{C}$ be a continuous complex function of a real variable t, $z(t)$ is a **simple curve** or **Jordan curve** (path or curve) if $z(t)$ is one to one (the curve does not intersect itself). It is **closed** if $z(a) = z(b)$. Such a closed curve is positive oriented when it is in the counterclockwise direction.*

Definition 3. *A **contour** is a piecewise smooth simple curve.*

Remark: Sometime we may require a contour to be piecewise differentiable.

Definition 4. *Let f be piecewise continuous on a contour C represented by $z(t) : [a, b] \to \mathbb{C}$. The line integral (contour integral) of f along C is defined to be*

$$\int_C f(z)dz = \int_a^b f\big(z(t)\big)z'(t)dt$$

Definition 5.

$$\int_C f(z)|dz| = \int_a^b f(z(t))|z'(t)|dt$$

Proposition 1.

$$\left|\int_C f(z)dz\right| \leq \int_C |f(z)||dz|$$

Example 1. *Evaluate the integral with the principal branch*

$$\int_C z^{-1+i}dz$$

where C is the positively oriented unit circle.

To parametrize a circle, we have $z(\theta) = e^{i\theta}$ for $\theta \in (-\pi, \pi]$.

$$f(z)dz = f(z(\theta))ie^{i\theta}d\theta = e^{(-1+i)i\theta}ie^{i\theta}d\theta = e^{-\theta}id\theta,$$

$$\int_C z^{-1+i}dz = \int_{-\pi}^{\pi} e^{-\theta}id\theta = i(-e^{-\pi} + e^{\pi}).$$

We should be careful that z^{-1+i} is not defined on the branch cut $\{arg(z) = \pm\pi\}$ but z^{-1+i} is still piecewise continuous on C.

Definition 6. *Suppose C is a contour represented by $z(t) : [a, b] \to \mathbb{C}$, then the length of the contour is the integral*

$$L = \int_a^b |z'(t)|dt$$

In $\mathbb{R}^2$, the line integral may be independent of the path taken (only depend on the two ends of the path), we would wonder if it is true for contour integral in $\mathbb{C}$.

Theorem 1. *Suppose that $f(z)$ is continuous in a open connected set D. The following statements are equivalent*

- *f(z) has an antiderivative $F(z)$ throughout D $\left(F'(z) = f(z)\right)$;*
- *Given any two fixed points z_1 and z_2 in D, for any contour lying in D with end points z_1 and z_2, the contour integral has a fixed value depends only on z_1 and z_2 (path independent);*
- *the contour integrals of $f(z)$ along any closed contours lying entirely in D all have value zero.*

Moreover, if $f(z)$ has an antiderivative $F(z)$, then

$$\int_C f(z)dz = \int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$$

Remark : We should consider $f(z) = 1/z$. It seems that the antiderivative of f is $F(z) = \log z$, however $\log z$ is not well-defined on the branch cut (in the principal branch, $\log z$ is not well-defined at the ray $arg(z) = \pm\pi$ and it can not be differentiable there). Hence $f(z) = 1/z$ does not have antiderivative in D. You can compute directly that $\int_{|z|=1} \frac{dz}{z} = 2\pi i$ which is not zero.

Exercise:

1. Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant, show that $\left|\int_C \frac{z+5}{z^2-1} dz\right| \leq 7\pi/3$.

2. Let C be the of the circle $|z-1| = 2$, compute $\displaystyle\int_C \frac{zdz}{z-1}$.

3. Compute the integral $\int_C f(z)dz$ with
(a) C is the arc of the semicircle $z = 2e^{i\theta}$ $(0 \leq \theta \leq \pi)$ and $f(z) = \dfrac{z+2}{z}$
(b) C consists of the arc of the semicircle $z = 1 + e^{i\theta}$ $(\pi \leq \theta \leq 2\pi)$ and the line segment $z = x$ with $x \in [0, 2]$. $f(z) = z - 1$.