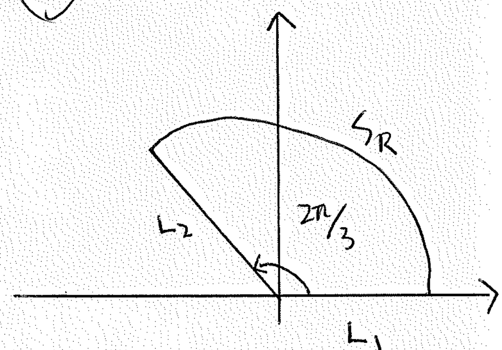
① we first consider the problem.

$\int_0^R \frac{dx}{1+x^3}$, the integral from 0 to R is related to the real line segment 0 to R in the complex plane. To make the line segment to become a closed contour, we join it by a part of circular arc and a straight line segment.

We first consider

$$\int_{C_R} \frac{dz}{1+z^3} = \int_{L_1} + \int_{L_2} + \int_{S_R} \frac{dz}{1+z^3} \quad - \text{(A)}$$



$$\int_{L_1} \frac{dz}{1+z^3} = \int_0^R \frac{dx}{1+x^3}$$

$$\int_{L_2} \frac{dz}{1+z^3} = \int_R^0 \frac{dx}{1+x^3} \cdot e^{2\pi i/3}$$

This is why we choose the angle $2\pi/3$ for the line segment L_2, it make $\left(x e^{2\pi i/3}\right)^3$ becomes x^3 such that the part for L_1 and L_2 can be put together.

$$\left| \int_{SR} \frac{dz}{1+z^3} \right| \leq \frac{2\pi R}{R^3 - 1} \quad \text{for } R \text{ large enough.}$$

observe that

$$(1+z^3) = (z+1)\left(z - e^{\pi/3 i}\right)\left(z - e^{-\pi/3 i}\right)$$

and $z = e^{\pi/3 i}$ is the singularity inside CR.

Thus by Cauchy integral formula,

$$\int_{CR} \frac{dz}{1+z^3} = 2\pi i \left(\frac{1}{e^{\pi/3 i}+1}\right)\left(\frac{1}{e^{\pi/3 i} - e^{-\pi/3 i}}\right)$$

By (A), $\left(1 - e^{2\pi/3 i}\right) \int_0^R \frac{dx}{1+x^3} + \int_{SR} \frac{dz}{1+z^3}$

$$= 2\pi i \left(\frac{1}{e^{\pi/3 i}+1}\right) \frac{1}{2i \sin \pi/3}$$

$$\Rightarrow \int_0^\infty \frac{dx}{1+x^3} = \frac{2\pi}{3\sqrt{3}} \quad \text{by taking } R \to \infty.$$

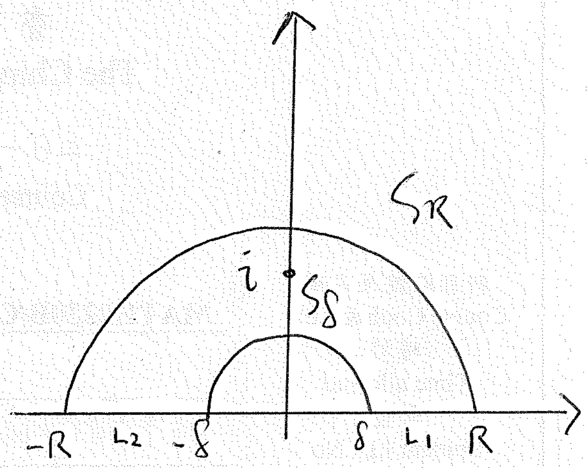
(2) We first consider the problem for large R and small δ,

$\int_\delta^R \frac{(\log x)^2}{1+x^2} dx$, since $\log x$ has singularity near $x=0$, The integral from δ to R is related to the real line segment δ to R in complex plane.

We complete the line segment by joining a semicircle. We consider

$$\int_C \frac{\left(\log z - i\pi/2\right)^2}{1+z^2} = \int_C f$$

$$= \int_{L_1} + \int_{L_2} + \int_{S_R} + \int_{S_\delta} \frac{\left(\log z - i\pi/2\right)^2}{1+z^2}.$$



This choice of f is not obvious, we will explain it in the integral L_1 and L_2.

$$\int_{L_1} = \int_\delta^R \frac{\left(\log x - i\pi/2\right)^2}{1+x^2} = \int_\delta^R \frac{(\log x)^2 - i\pi\log x - \left(\frac{\pi}{2}\right)^2}{1+x^2}$$

$$\int_{L_2} = \int_{-R}^{-\delta} \frac{\left(\log x - i\pi/2\right)^2}{1+x^2}$$

$$= \int_\delta^R \frac{\left(\log(-x) - i\pi/2\right)^2}{1+x^2}$$

$$= \int_\delta^R \frac{\left(\log xe^{i\pi} - i\pi/2\right)^2}{1+x^2}$$

$$= \int_\delta^R \frac{\left(\log x + i\pi/2\right)^2}{1+x^2}$$

$$= \int_\delta^R \frac{(\log x)^2 + i\pi\log x - \left(\pi/2\right)^2}{1+x^2}$$

$$\left|\int_{S_R} \frac{\left(\log z - i\pi/2\right)^2}{1+z^2}\right| \leq 2\pi R \cdot \frac{\log^2 R + \left(\pi/2\right)^2}{R^2-1} \quad - (A)$$

$$\left|\int_{S_\delta} \frac{\left(\log z - i\pi/2\right)^2}{1+z^2}\right| \leq 2\pi \delta \cdot \frac{\log^2 \delta + \left(\pi/2\right)^2}{\delta^2-1} \quad - (B)$$

$$\int_C \frac{\left(\log z - i\pi/2\right)^2}{1+z^2} dz = \int_C \frac{\left(\log z - i\pi/2\right)^2}{(z-i)(z+i)}$$

$$= 2\pi i \frac{\left(\log i - i\pi/2\right)^2}{2i}$$

$$= 0$$

Thus $0 = \int_C \frac{\left(\log z - i\pi/2\right)^2}{1+z^2} = \int_{L_1} + \int_{L_2} + \int_{S_R} + \int_{S_\delta} \frac{\left(\log z - i\pi/2\right)^2}{1+z^2}$

$$0 = 2\int_\delta^R \frac{(\log x)^2}{1+x^2} - \frac{\pi^2}{2}\int_\delta^R \frac{1}{1+x^2} + \int_{S_R} + \int_{S_\delta} \frac{\left(\log z - i\pi/2\right)^2}{1+z^2}$$

(This explain the choise of f)

By (A) and (B), we take $R \to \infty$, $\delta \to 0$.

$$0 = 2\int_0^\infty \frac{(\log x)^2}{1+x^2} - \frac{\pi^2}{2}\int_0^\infty \frac{1}{1+x^2}$$

Since $\int_0^\infty \frac{1}{1+x^2} = \left[\tan^{-1} x\right]_0^\infty = \pi/2$

Thus $\int_0^\infty \frac{(\log x)^2}{1+x^2} = \frac{\pi^3}{8}$.

③ We can carry out partial fraction such that $\dfrac{2a}{a^2+\sin^2\theta} = \dfrac{1}{a+i\sin\theta} + \dfrac{1}{a-i\sin\theta}$ to deal with the square term $\sin^2\theta$. But the computation is more complicated since we produce 2 terms from 1 term. We use the double angle formula such that $a^2+\sin^2\theta = a^2 + \dfrac{1-\cos 2\theta}{2}$

$$= \frac{1}{2}\left(2a^2+1-\cos 2\theta\right)$$

$$\int_0^{\pi} \frac{d\theta}{a^2+\sin^2\theta} = \int_0^{2\pi} \frac{d\theta}{2a^2+1-\cos\theta}$$

The range of integral 0 to 2π is related to a unit circle. For $z=e^{i\theta}$,

$$\int_0^{\pi} \frac{d\theta}{a^2+\sin^2\theta} = \int_0^{2\pi} \frac{d\theta}{2a^2+1-\cos\theta}$$

$$= \int_0^{2\pi} \left(\frac{1}{iz}\right) \frac{d(e^{i\theta})}{2a^2+1-\frac{1}{2}\left(z+\frac{1}{z}\right)}$$

$$= \int_C \frac{dz}{z^2-(2+4a^2)z+1}$$ where C is the unit circle.

(3) $a^2+\sin^2\theta = a^2 + \dfrac{1-\cos 2\theta}{2}$

$$= \frac{1}{2}\left(2a^2+1-\cos 2\theta\right)$$

$$\int_0^{\pi} \frac{d\theta}{a^2+\sin^2\theta} = \int_0^{2\pi} \frac{d\theta}{2a^2+1-\cos\theta}$$

Consider $f(z) = \dfrac{1}{iz}\left(\dfrac{1}{1+2a^2-\frac{1}{2}\left(z+\frac{1}{z}\right)}\right)$

$$= \frac{2i}{z^2-(2+4a^2)z+1} \quad \text{for } |z|=1$$

Roots of $z^2-(2+4a^2)z+1=0$ are

$$z_1 = \frac{2+4a^2+\sqrt{16a^2+16a^4}}{2} = 1+2a^2+2|a|\sqrt{1+a^2}$$

and $\quad z_2 = 1+2a^2-2|a|\left(\sqrt{1+a^2}\right)$

z_2 is inside the unit circle, then

$$\int_C f(z) = -2\pi i \, \frac{2i}{z_1-z_2} = \frac{\pi}{|a|\sqrt{1+a^2}}$$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{2a^2+1-\cos\theta} = \frac{\pi}{|a|\sqrt{1+a^2}}$$

Root of $z^2-(2+4a^2)z+1$ are

$$z_1 = \frac{2+4a^2+\sqrt{16a^2+16a^4}}{2} = 1+2a^2+2|a|\sqrt{1+a^2}$$

and $z_2 = 1+2a^2-2|a|\sqrt{1+a^2}$. Since z_2 is inside C, by Cauchy Integral formula,

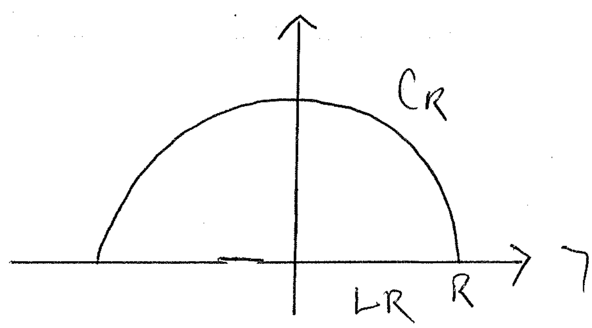
$$\int_0^{\pi} \frac{d\theta}{a^2+\sin^2\theta} = \int_C \frac{dz}{(z-z_1)(z-z_2)}$$

$$= 2\pi i \frac{1}{z_2-z_1}$$

$$= \frac{\pi}{|a|\sqrt{1+a^2}}.$$

(4) we first consider $\int_{-R}^{R} \frac{\sin x}{x^2+4x+5}$. The integral from $-R$ to R is related to the real line segment $-R$ to R in the complex plane. Then we complete the line by joining a semicircle of radius R.

we consider $\int_C \frac{e^{iz}}{z^2+4z+5} = \int_{C_R} + \int_{L_R} \frac{e^{iz}}{z^2+4z+5}$



$$\left| \int_{C_R} \frac{e^{iz}}{z^2+4z+5} \right| \le \frac{\pi}{(R-\sqrt{5})^2}$$ by Jordon Lemma

$$\int_{L_R} \frac{e^{iz}}{z^2+4z+5} = \int_{-R}^{R} \frac{\cos x + i \sin x}{x^2+4x+5}$$

$$\int_C \frac{e^{iz}}{z^2+4z+5} = \int_C \frac{e^{iz}}{(z-(-2-i))(z-(-2+i))}$$

$$= 2\pi i \left(\frac{e^{i(-2+i)}}{2i} \right)$$

$$= \pi e^{-1-2i}$$

Thus $\displaystyle\int_{-R}^{R} \frac{\cos x + i \sin x}{x^2+4x+5} + \int_{C_R} \frac{e^{iz}}{z^2+4z+5} = \int_C \frac{e^{iz}}{z^2+4z+5}$

$$= \pi e^{-1-2i}$$

Take $R \to \infty$ and compare the imaginary part of both side,

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5} = -\frac{\pi \sin 2}{e}.$$