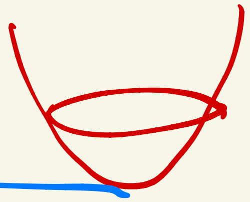
Q: How to determine M using Rm??

Thm (Cartan-Hadamard) If (M,g) is complete simply connected mfd with $K_g \le 0$, then M is diffeomorphic to $\mathbb{R}^n$.



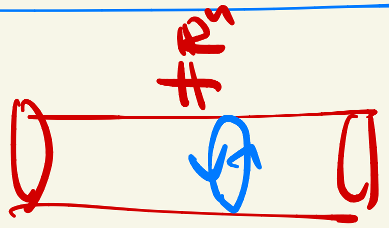
Idea: $\exp_p : \overset{\cong \mathbb{R}^n}{T_pM} \longrightarrow M.$ is a diff. ??

By completeness, this is surjective

issue (i) : one-one ?? ☆☆

(ii) : differentiable ?? ☆☆

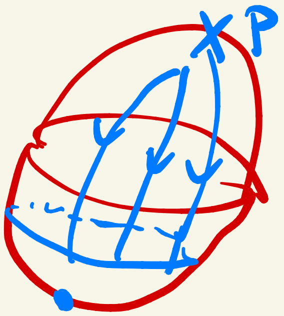
trivial example: ①



flat cylinder $K_g \equiv 0$.

that's why we need simply connected!!

②



Lemma: Under $K \leq 0$, there is No conjugate pt of p, hence $\exp_p$ is a local diffeomorphism.

pf: Suppose Not. $\exists\, \gamma : [a,b] \to M$ geod. from $\gamma(a) = p$, $\gamma(b) = q$ s.t. we can find a Jacobi field J s.t. $J(a) \overset{=0}{=} J(b)$, $J \not\equiv 0$.

let $f(t) = \frac{1}{2} |J(t)|^2$. s.t.

$$f' = \langle J, J' \rangle \quad \text{and} \quad f'' = \overbrace{\langle J' J' \rangle}^{\geq 0} + \underbrace{\langle J, J'' \rangle}_{\underset{\sim\sim\sim}{- R(\gamma', J, J, \gamma') \geq 0}}$$

$\Rightarrow f''$ is convex.

$\Rightarrow f'(t) \geq f'(a) = 0 \Rightarrow f \uparrow$

$\Rightarrow |J(t)|^2 \geq |J(a)|^2 = 0$.

$\Rightarrow J(t) \equiv 0$ since $\left| \begin{array}{l} J(b) = 0 \\ f \text{ is convex.} \end{array} \right.$

$\Rightarrow \rightarrow\leftarrow$. #.

Lemma: Suppose $F: M^n \to N^n$ is a local isometry ($\forall p \in M$, $\exists U$ s.t. $F|_U$ is iso.),

then ① F is distance decreasing.

② $F \circ \gamma(t)$ is geod. if $\gamma(t)$ is geod.

③ $F \circ \exp_p(v) = \exp_{F(p)} \circ dF|_p(v)$, $\forall v \in T_pM$ well-def.

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pf. ①: let $p, q \in M$, let $\gamma(t)$ be geodesic from p to q on M.

$$d_N(F(p), F(q)) \leq L_N(F \circ \gamma(t))$$

$$= \int_a^b |(F \circ \gamma)'| \, dt$$

$$\boxed{\text{local isometry} \Rightarrow \langle u, v \rangle = \langle dF(u), dF(v) \rangle}$$

$$= \int_a^b |dF(\gamma')|_N \, dt$$

$$= \int_a^b |\gamma'|_M \, dt$$

$$= d_M(p,q). \quad \#$$

② $F \circ \gamma(t) = \tilde{\gamma}(t)$ = geodesic.

follows from $dF\big(\underset{\text{connection of } F^*g}{\nabla_u} v\big) = \overset{\text{connection of } g}{\tilde{\nabla}}_{dF(u)} dF(v).$

③ $F \circ \exp_p(v) = \exp_{F(p)} \circ dF|_p(v).$

pf: let $\gamma(t) = F \circ \exp_p(tv)$

$= \text{geod. from } F(p) = \gamma(0).$

w/

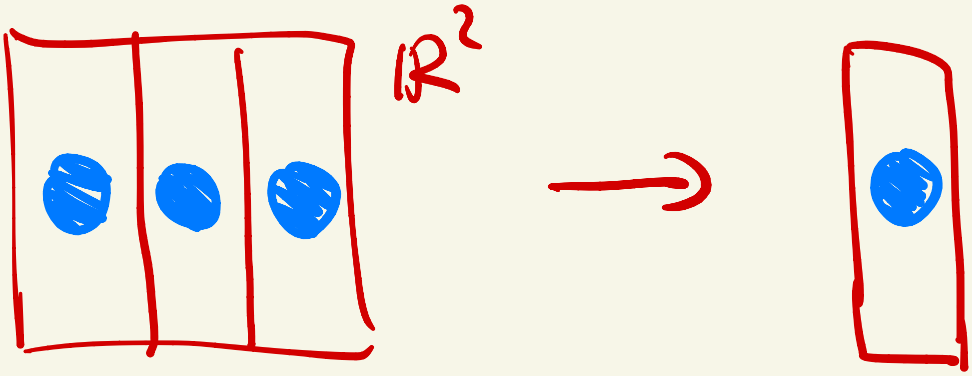
$\gamma'(0) = dF|_p(v).$

$\overset{\text{uniqueness}}{=\!=} \exp_{F(p)}\big(t \cdot dF|_p(v)\big). \quad \#$

$\boxed{\text{taking } t=1}$

Lemma: If $F: M^n \to N^n$ is a local isometry, N is complete, then $\forall p \in N$, $\exists\, \mathcal{U}$ s.t. $F^{-1}(\mathcal{U}) = \coprod_{\alpha} \mathcal{U}_\alpha$ and $\forall \alpha$, $F|_{\mathcal{U}_\alpha} : \mathcal{U}_\alpha \to \mathcal{U}$ is a diffeomorphism

Example:



Pf: Let $p \in N$, take $r \ll 1$ s.t.

$\mathcal{U} = B(p, r) \subset$ Normal coordinate at p.

Let $F^{-1}(p) = \{p_\alpha\}$

Denote $\mathcal{U}_\alpha = B(p_\alpha, r)$

Claim: $F|_{\mathcal{U}_\alpha} : \mathcal{U}_\alpha \to \mathcal{U}$ is diff.

pf: Step 1: $F(\mathcal{U}_\alpha) \subseteq \mathcal{U}$.

let $s > 0$ be sup s.t. $F(B(p_\alpha, s)) \subseteq \mathcal{U}$.

then $s = r$. exists

(If Not, $\exists\, x \in \overline{B(p_\alpha, s)}$ s.t.

$$d(F(x), F(p_\alpha)) = r$$

$$\wedge$$

$$d_M(x, p_\alpha) \leq s \quad \# .)$$

$\therefore F(\mathcal{U}_\alpha) \subseteq \mathcal{U}$.

Step 2: $\mathcal{U} \subseteq F(\mathcal{U}_\alpha)$

Let $y \in B(p, r) = \mathcal{U}$.

$\Rightarrow y = \exp_p(v)$ for some $v \in T_pM$ w/ $|v| < r$.

$\therefore \exists\, u \in T_{p_\alpha}M$ s.t. $dF|_{p_\alpha}(u) = v$.

w/ $|u| = |v| < r$ (local iso.)

$$\Rightarrow y = \exp_p \circ dF|_p(u)$$

$$\overset{\text{uniq. of geo.}}{=} F \circ \underbrace{\exp_{p_\alpha}(u)}_{M} \quad \#.$$

<u>step 3</u>: injective since if

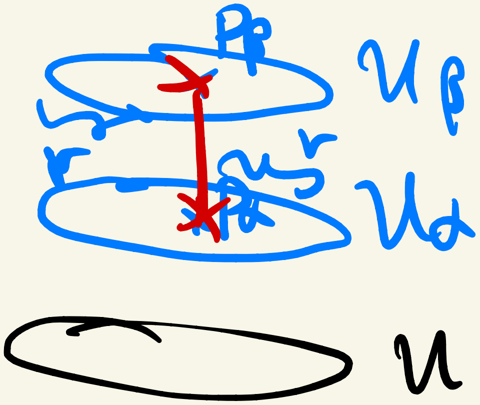
$F(x) = F(y)$ for some $x, y \in U_\alpha$.

then $x = y$ since $\exp \circ dF = F \circ \exp_{p_\alpha}$

$\#$

<u>step 4</u>: $U_\alpha \cap U_\beta = \phi$ if $\alpha \neq \beta$.

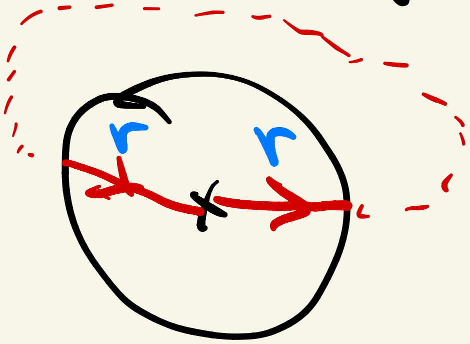
Pf:



Claim: $d(p_\alpha, p_\beta) \geq 2r$

otherwise take γ from p_α to p_β.

$F\circ\gamma$ = geodesic loop of p.



$\Downarrow$

$L(F\circ\gamma) \geq 2r$

which contradicts with

$$L(\gamma) = d(p_\alpha, p_\beta) < 2r.$$

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pf of Cartan Hadamard thm:

$$F: \overset{\cong N}{T_pM} \to M \text{ by } F(X) = \exp_p(X).$$

- lemma $\Rightarrow$ F is a local diff.

Equip N with metric $h = F^*g$.

- $F: (T_pM, h) \longrightarrow (M, g)$, this is local isometry,

- (T_pM, h) is a complete Riemannian mfd

pf: Suffices to prove it is complete metric space.

Let $x_i \to +\infty$ on $N = T_pM$.

then $$d_h(x_i, 0) \overset{\text{distance decreasing}}{\geq} d_g(\exp_p(x_i), p) = |x_i| \to +\infty$$

$\therefore$ (T_pM, h) is complete.

lemma.

$\Rightarrow$ $\{$ $F: N \to M$ is a covering map

$\lfloor$ M = $\underline{\text{simply connected}}$

$\Rightarrow \forall q \in M,\ F^{-1}(q) = \{q\}$

$\therefore$ F is a diffeomorphism. #

Rmk: If $M \neq$ simply connected,

then consider the universal cover!!

Q: What if $\boxed{K=0}$, then can we

improve diffeomorphism to isometry??

Thm (Cartan) If (M,g) is complete, simply connected with $K_g = \left\{ \begin{matrix} 1 \\ 0 \\ -1 \end{matrix} \right.$ (either)

then M is isometric to

$$K_g = \begin{cases} 1 & \text{to sphere} \\ 0 & \text{to Euclidean} \\ -1 & \text{to hyperbolic.} \end{cases}$$

Rmk: by scaling, it work for all constant sectional curvature.

Main problem ① find diffeomorphism

② prove diff preserve distance

Easy if $M \cong \mathbb{R}^n$ topologically $(K_g \leq 0)$

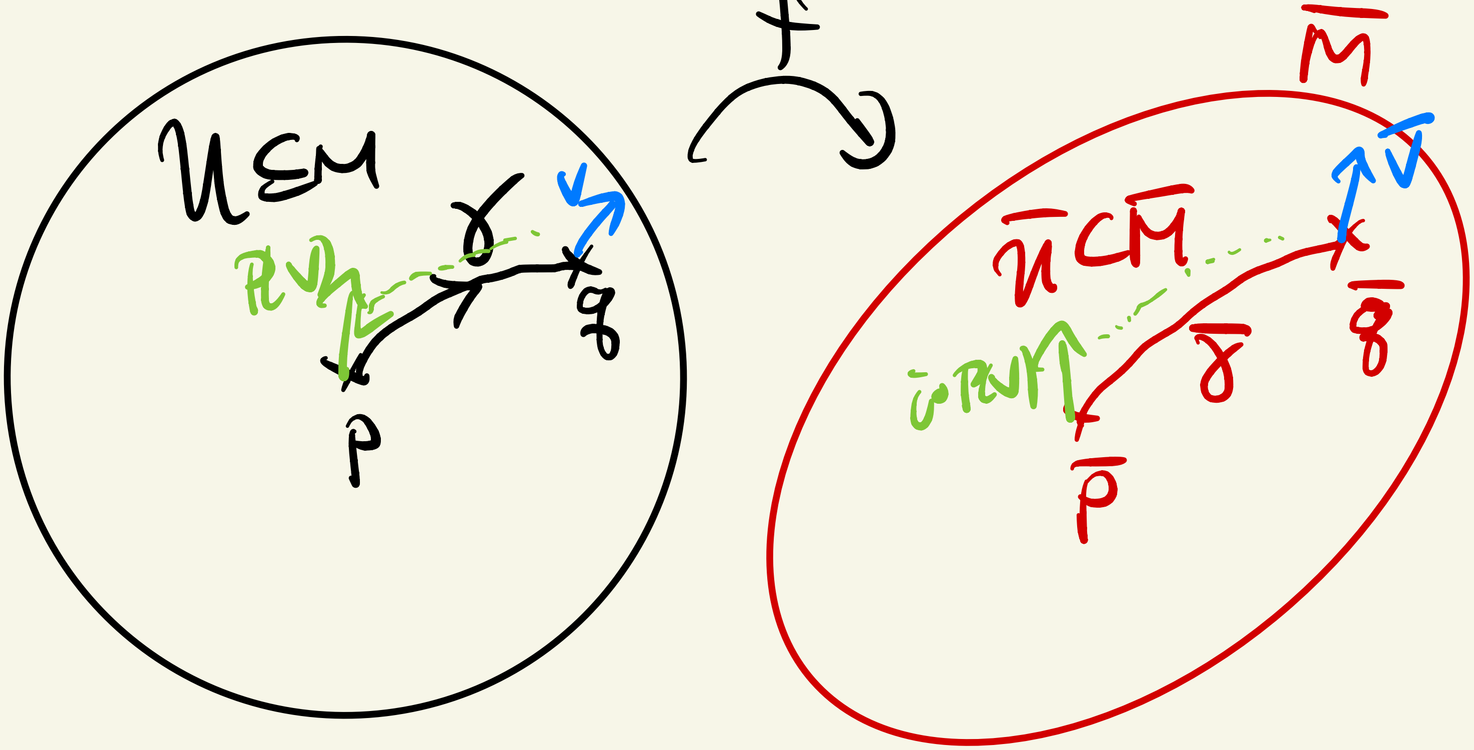
Construct diffeomorphism :

spaceform

$p \in M$, $\bar{p} \in \bar{M}$

T_pM $\quad$ $T_{\bar{p}}\bar{M}$

Let $i : T_pM \longrightarrow T_{\bar{p}}\bar{M}$ be the isometry (which map O.N. basis to ON basis)



Define $f(q) = \exp_{\bar{p}} \circ\, i \circ \exp_p^{-1}(q)$.

for $q \in$ Normal nbd of p.

$$f : \mathcal{U} \to \overline{\mathcal{U}} = f(\mathcal{U})$$

Define : for $v \in T_qM$,

$$\bar{v} = \overline{P}_t \circ i \circ P_t^{-1}(v) = \phi(v)$$

where P_t is the parallel transport from T_pM to T_qM along geodesic γ : p to q

And $\overline{P}_t$ is the parallel transport from $\overline{T}_pM$ to $\overline{T}_qM$ along the geodesic $\bar\gamma$ given by $\begin{cases} \bar\gamma(0) = \bar p \\ \bar\gamma'(0) = i(\gamma'(0)) \end{cases}$

Thm: If $\forall\, q \in \mathcal{U}$, $\forall\, x, y, u, v \in T_qM$

we have "$Rm = (4,0)$ type tensor."

$$R(x,y,u,v) = \bar{R}(\phi(x), \phi(y), \phi(u), \phi(v))$$

$\Updownarrow$ determine

K_q

then f = Isometry locally

and $df = i$.

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pf: It suffices to show that

$$|v| = |df(v)|, \quad \forall\, v \in T_qM,\ q \in \mathcal{U}.$$

since $\langle v, v \rangle = \langle df(v), df(v) \rangle \quad \forall\, v$.

differentiate s.t. $v' = u$ yields

$$2\langle v, u \rangle = 2\langle df(v), df(u) \rangle.$$

$\Rightarrow$ f is local Isometry : (preserve inner product)

· Let J be the Jacobi field s.t. $\begin{cases} J(0) = 0 \\ J(l) = v \end{cases}$

where $\gamma : [0, l] \to M$ a geod from p to q.

· $\bar{J}(t) \triangleq \phi(J(t))$ where $\phi = \bar{P}_t \circ i \circ P_t^{-1}$

claim: $\bar{J}$ is a Jacobi field.

pf: • $J(t) = \sum X_i(t) \cdot e_i(t)$ where

$\{e_i(t)\}$ is parallel o.n. frame along γ.

Jacobi field $\Leftrightarrow$ $\forall j$, $X_j'' + \sum_{i=1}^{n} R(\gamma', e_i, \gamma', e_j) x_j = 0$

• $\bar{J}(t) = \phi(J(t)) = \sum_{i=1}^{n} X_i(t)\, \phi(e_i(t))$

where $\{\phi(e_i(t)) = \bar{e}_i(t)\}$ is parallel o.n.

along $\bar{\gamma}$.

$\Rightarrow$ $\bar{J}$ satisfies $X_j'' + \sum_{i=1}^{n} R(\gamma', e_i, \gamma', e_j) x_j = 0$

$\parallel$ (assumption)

$\bar{R}(\phi(\gamma'), \phi(e_i), \phi(\gamma'), \phi(e_j))$

∴ $\bar{J}$ also satisfies Jacobi field equation. #

And $\bar{J}(0) = 0$ since $x_i(0) = 0$ $\forall i$.

And $|\bar{J}| = \sqrt{\sum x_i^2} = |J|$.

$f = \exp_{\bar{p}} \circ i \circ \exp_p^{-1}$

$$\therefore |v| = |\bar{J}(\ell)| \stackrel{?}{=} |df(v)|$$

with

$$\bar{J}'(0) = \sum_{i=1}^{n} x_i'(0)\, \bar{e}_i(0)$$

$$= i\left(\sum_{i=1}^{n} x_i'(0)\, e_i(0)\right)$$

$$= i(J'(0))$$

uniqueness

$$\bar{J}(\ell) = d\exp_{\bar{p}}\big|_{\ell\bar{\gamma}'}\left(\ell \cdot i(J'(0))\right)$$

$$= d\exp_{\bar{p}}\big|_{\ell\bar{\gamma}'} \circ i \circ \left(d\exp_p\big|_{\ell\gamma'}\right)^{-1}(J(\ell))$$

$$= df\big|_q(v). \quad \#$$

pf of Cartan thm for $K_g = 0$ or -1.

- By Cartan-Hadamard thm (or its proof),

$$f : \underset{K=0}{\mathbb{R}^n} / \underset{K=-1}{\mathbb{H}^n} \to M \text{ given by}$$

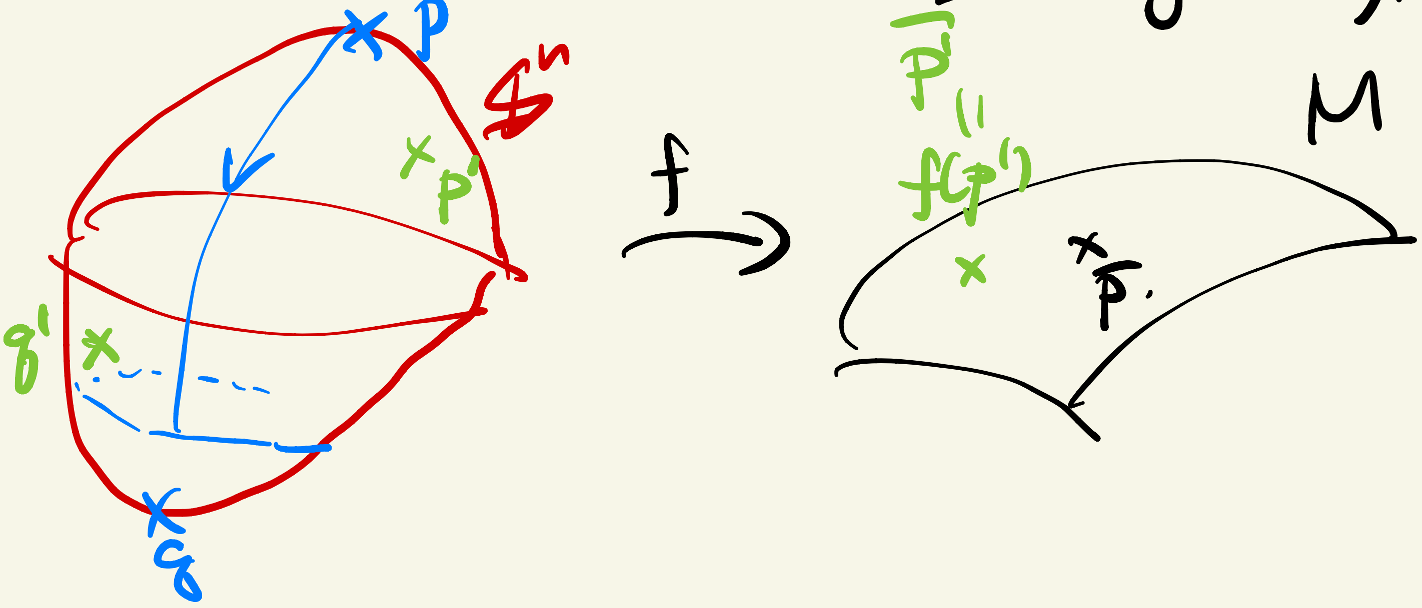
$$f(x) = \exp_p \circ i \circ \exp_{\bar{p}}^{-1} \text{ is well-defined } \overset{(\text{diff.})}{} \text{ globally.}$$

Since $K = c$, $\Rightarrow R_{ijkl} = \pm c(g_{ik}g_{jl} - g_{il}g_{jk})$

(Above thm assumption ✓)

$\Rightarrow f = $ local isometry $\Longrightarrow$ isometry #.

Remains to check the case of S^n. ($K_g = 1$).



Define $f : \exp_{\bar{p}} \circ i \circ \exp_p^{-1} : S^n \setminus \{q\} \to M$.

- still have $f =$ local isometry

- Goal: wish to extend f across q.

Wish: Find a new $f' : S^n \setminus \{p'\} \to M$

s.t. $f' = f$ away from p', p

pick p' $\neq p, q$. in a way that

- $f'(p') = \bar{p} \overset{\text{chosen}}{=} f(p')$
- $df'|_{p'} = \bar{i} \overset{\text{chosen}}{=} df|_{p'}$

By Applying the last thm

$\Rightarrow$ $f' =$ local isometry satisfying

$\therefore \exists f : S^n \setminus \{q\} \to M$

$f' : S^n \setminus \{q'\} \Rightarrow M$ which are local isometry w/ $q \neq q'$. And

$$\boxed{f \equiv f' \text{ on } S^n \setminus \{q, q'\}}$$ Why?!

Define $F : S^n \Rightarrow M$ by

$$F = \begin{cases} f & \text{on } S^n \setminus \{q\} \\ f' & \text{on } S^n \setminus \{q'\} \end{cases}$$ is a local isometry.

$\Rightarrow$ F = covering map

$\Rightarrow$ F = diffeomorphism by simply connected

\#

Suffice to show that

$A = \{ x \in S^n \setminus \{q, q'\} : f' = f, df = df' \}$

is open + closed ✓. $\Rightarrow$ $A = S^n \setminus \{q, q'\}$

By $f \circ \exp_x(v) = \exp_{f(x)} \circ df|_x(v)$. #