Recall:

Thm (Meyer): If (M, g) is complete mfd with $Ric \geq (n-1)k$ $(k>0)$, then M is cpt and

$$\operatorname{diam}(M, g) \leq \frac{\pi}{\sqrt{k}}.$$

pf: By 2nd variational formula of length.

More generally, how Ricci curvature restrict the geometry?

Volume comparison thm: Suppose (M, g) is complete Riemannian mfd with $Ric \geq \underline{(n-1)k}$, ($k>0$, $=0$, or <0)

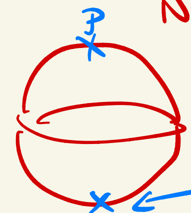
Ricci curvature of spaceform

then

$$\forall p \in M, \quad \frac{Vol(B(p,r))}{vol_k(\bar{B}_k(r))} \text{ is non-increasing in } r>0.$$

- $\partial B(p,r) = \{x \in M : \boxed{d(x,p)} = r\}$ a function on M.
- $B(p,r) = \{x \in M : d(x,p) < r\}$

Not necessarily smooth



Some issue here.

"Set of issue":

complete mfd

Given a geodesic $\gamma : [0,+\infty) \longrightarrow M$.

$$S_\gamma = \{ t \in [0,+\infty) : d(\gamma(0), \gamma(t)) = t \}$$

$\Rightarrow S_\gamma$ is either $\underbrace{[0,+\infty)}_{\text{good case}}$ or $\underbrace{[0, t_\gamma]}_{\text{Bad case}}$

$\gamma(t_\gamma)$ = cut point of $\underset{\gamma(0)}{\overset{\|}{p}}$ along γ.

Defn: $\text{cut}(p) = \{ \gamma(t_\gamma) \mid \gamma : [0,+\infty) \to M$ is geod. w/ $\gamma(0) = p \}$

$\text{cut}(p)$ $\|$ cut locus of p

$\boxed{\text{Rmk: If } M \text{ is cpt, then } \text{cut}(p) \neq \phi \ \forall p \in M.}$

$$\text{Seg}(p) = \{ v \in T_pM : \underbrace{\exp_p(tv) \text{ is a segment}}_{\Updownarrow\ d(\gamma(0),\gamma(t)) = t} \text{ on } [0,1] \}$$

$$\text{seg}^\circ(p) = \{ sv \in T_pM : s \in [0,1), \ v \in \text{seg}(p) \}$$

then $\begin{cases} \text{cut}(p) = \exp_p\left(\text{seg}(p) - \text{seg}^\circ(p)\right), \\ M = \exp_p(\text{seg}(p)) \text{ by completeness of } M. \end{cases}$

---

prop: Given $p \in M$ and $\gamma : [0, t_0) \to M$, geod. from p,

then $\gamma(t_0)$ is a cut pt of $\gamma(0) = p$ along γ

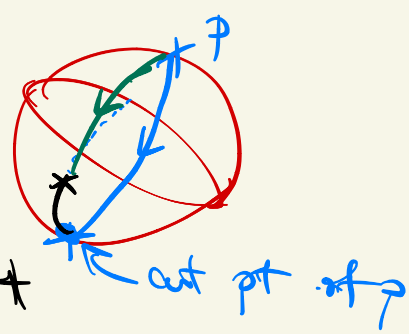
iff one of the following holds at $t = t_0$

and neither of them holds before t_0.

(i) $\gamma(t_0)$ is a conjugate pt of $\gamma(0) = p$

(ii) $\exists$ geod. $\sigma \neq \gamma$ from $\gamma(0) = \sigma(0) = p$ to $\gamma(t_0)$ s.t.

$$L(\sigma) = L(\gamma).$$



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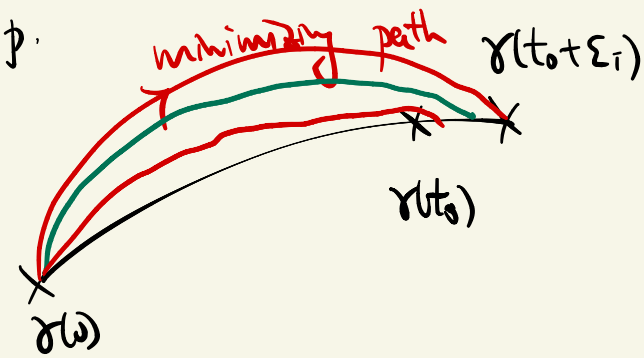
pf: $(\Rightarrow)$ Suppose $\gamma(t_0)$ is a cut pt

of $\gamma(0) = p$.

take $\varepsilon_i \downarrow 0$.



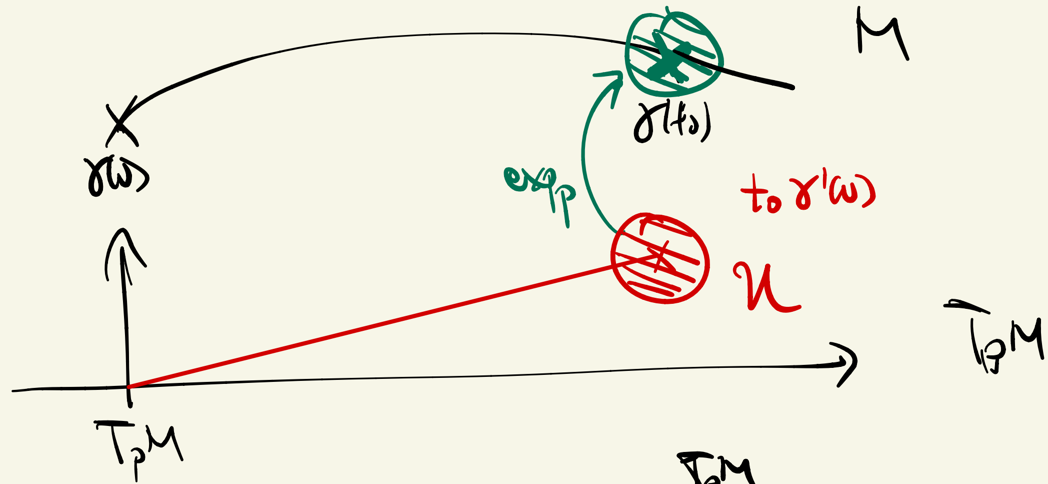
By completeness, $\exists\ \sigma_i$ minimizing geod. from $\gamma(0)$ to $\gamma(t_0+\varepsilon_i)$

By passing to subseq, $\sigma_i \to \sigma_\infty$: minimizing geod.

from p to $\gamma(t_0)$

Ⓐ $\sigma_\infty \neq \gamma$ then (ii) holds. since σ_∞ is minimizing.

(B) $\sigma_\infty = \gamma \ni t$: $d\exp_p$ is singular at $t_0\gamma'(0)$.



$$\begin{matrix} T_pM \\ \cup\!\!| \end{matrix}$$

If $d\exp_p$ is non-singular, then $\exists\ \mathcal{U} \ni t_0\gamma'(0)$ s.t.

$d\exp_p\big|_{\mathcal{U}}$ is diffeomorphism onto the image.

(σ_i is minimizing) $\Rightarrow$ $\gamma(t_0+\varepsilon_i) = \sigma_i(t_0+\varepsilon_i')$ for some $\varepsilon_i' \le \varepsilon_i$

and $\forall\, i\to\infty$, $\gamma(t_0+\varepsilon_i) \in d\exp_p(\mathcal{U})$.

$$\Rightarrow \quad d\exp_p\left[(t_0+\varepsilon_i)\gamma'(0)\right] = \gamma(t_0+\varepsilon_i)$$

$$= \sigma_i(t_0+\varepsilon_i')$$

$$= d\exp_p\left((t_0+\varepsilon_i')\,\sigma_i'(0)\right)$$

($\exp_p$ is diff on $\mathcal{U}$)

$$\Rightarrow (t_0+\varepsilon_i)\,\gamma'(0) = (t_0+\varepsilon_i')\,\sigma_i'(0)$$

$$\Rightarrow \begin{cases} \gamma'(0) = \sigma_i'(0) \\ \varepsilon_i = \varepsilon_i' > 0. \end{cases} \quad \forall\, i\to\infty.$$

$\Rightarrow \quad t_0 \gamma'(0) \in \mathrm{Seg}^0(p)$

$\Rightarrow$ contradiction since σ_i is minimizing

But $\gamma(t_0)$ is cut pt. #

Conversely, if ①: $\gamma(t_0)$ is the first conjugate pt of $p = \gamma(0)$,

(Done bef. by Index Lma)

then geod. is Not minimizing beyond $\gamma(t_0)$

$\Rightarrow$ cut point occurs before or at t_0.

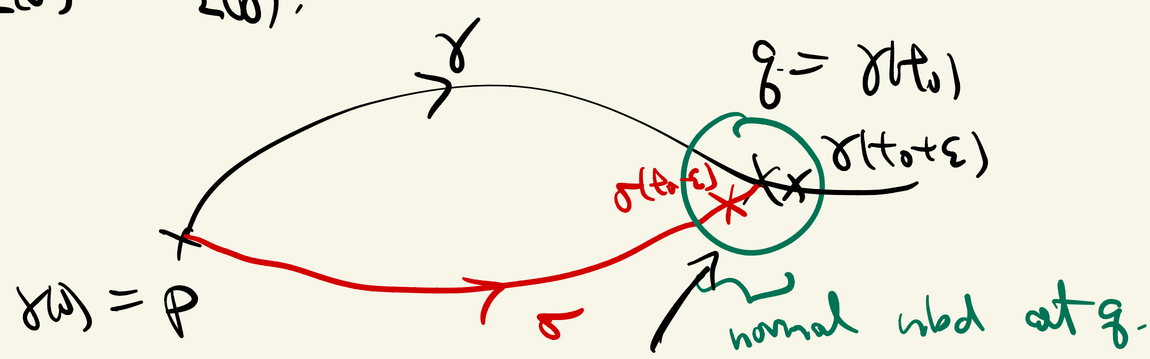
But if $\gamma(t_1)$ is cut. pt for some $t_1 < t_0$,

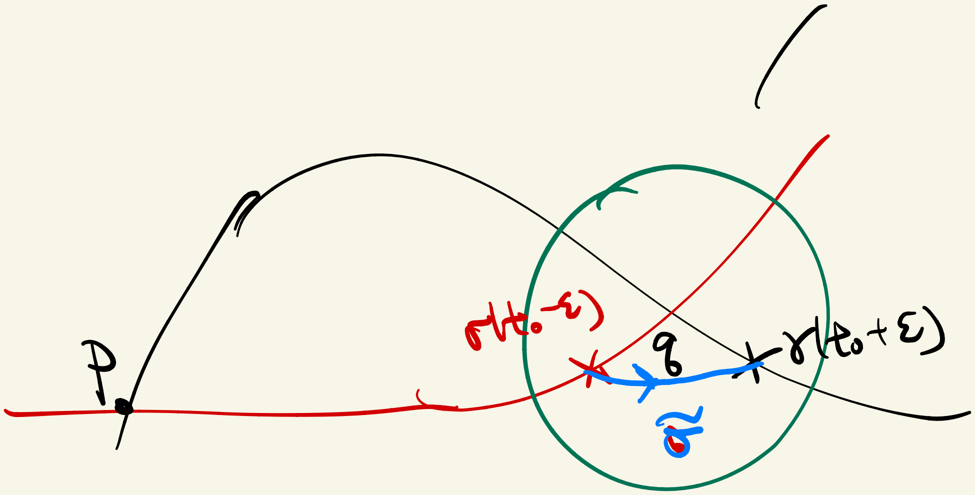
then ① or ② happened. which is Not true by assumption.

$\Rightarrow \gamma(t_0)$ is a cut pt.

If ② holds: $\exists\ \sigma \neq \gamma$ from $\gamma(0) = p$ to $\underset{\sigma(t_0)}{\overset{\shortparallel}{\gamma(t_0)}}$ s.t.

$L(\sigma) = L(\gamma)$.





Inside normal nbd of q, find the minimizing geodesic $\tilde{\sigma}$ from $\sigma(t_0-\varepsilon)$ to $\gamma(t_0+\varepsilon)$ for $0<\varepsilon\ll 1$.

then $\sigma\cup\tilde{\sigma}$ = geodesic from p to $\gamma(t_0+\varepsilon)$ with Length $< t_0+\varepsilon$ $(\sigma\neq\gamma)$

$\Rightarrow d(\gamma(t_0+\varepsilon),\gamma(0)) < t_0+\varepsilon$, $\forall\,\varepsilon>0$

$\Rightarrow$ cut point happens before $\gamma(t_0+\varepsilon)$, $\forall\,\varepsilon>0$

$\Rightarrow$ cut point happens before or at $\gamma(t_0)$.

Same argument in ① $\Rightarrow$ cut point at t_0

Conseq: ① $M=\exp_p(\mathrm{seg}(p))$ by completeness

② on $\exp_p(\mathrm{seg}^\circ(p))$, ← dense on M

may define $r(x) = |\exp_p^{-1}(x)| = d(x,p)$

where $r(\cdot)$ is smooth here.

* $|\nabla r| = 1$ on $\exp_p(\mathrm{seg}^\circ(p))$ by Gauss lemma.

$\left(g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1 \right)$

* on M globally, $r(\cdot)$ is Lipschitz fcn.

by triangle ineq.

Comparison thm:

$S^{n-1} \subset T_pM$.

In Normal polar coordinate (r, θ),

If $x = (r, \theta) \notin \mathrm{cut}(p)$, then

volume element $= J(\theta, r)\, dr \wedge d\theta$

$\left(\text{when } M = \mathbb{R}^n, \quad \bar{J}(\theta, r) = r^{n-1} \right)$

goal: compare $J(\theta, r)$ with $\begin{cases} \bar{J}(r). & (\text{if } Ric \geq 0) \\ \bar{J}_k(r) & \text{if } Ric \geq (n-1)k \end{cases}$

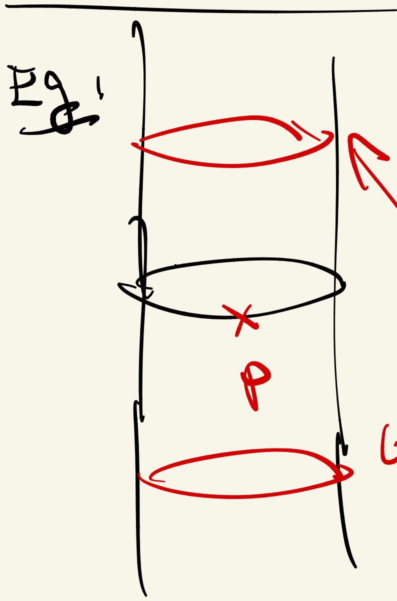
---

with this notation, we also know, by Gauss lemma.

for a.e. $r > 0$, $\quad |\partial B(r)| = \int_{C(r)} J(\theta, r)\, d\theta$

where $C(r) = \{ v \subset S_pM : \exp(sv) = \text{minimizing up to } r \}$

Eg:



$C(r) \subsetneq S_pM = \{ v \in T_pM : |v| = 1 \}$

$|\partial B(r)|$ if $r \gg 1$.

$(dA_t)' = H\, dA_t, \quad dA = J\, dr\, d\theta$

first variation formula

$\boxed{J \text{ v.s. } \bar{J}_k.}$

$J' = \partial_r J \overset{\text{first variation formula}}{=} H \cdot J$

$\Downarrow$

$J'' = \partial_r \partial_r J = H' J + H J'$

2nd variation formula. but keeping H $\quad = \; - \underbrace{|A|^2 J}_{\text{good}} - \underbrace{R_{rr} J}_{(n-1)k} + H^2 J$

g is locally Euclidean at p $(r=0)$ $\Rightarrow$

* $J \sim r^{n-1}$ as $r \to 0$
* $J' \sim (n-1) r^{n-2}$ as $r \to 0$

$$J'' \le \underbrace{-(n-1)k\,J}_{\text{Ricci estimate}} + H^2 J - \frac{1}{n-1} H^2 J$$

$$\le \frac{n-2}{n-1} H^2 J - (n-1)k\,J$$

$|A|^2 = \sum_{i,j=1}^{n-1} A_{ij}^2$, on "$S^{n-1}$"

$\ge \sum_{i=1}^{n-1} A_{ii}^2 \ge \frac{1}{n-1} H^2$

Cauchy ineq.

ODE:

$$\begin{cases} \bar{J}'' = -(n-1)k\,\bar{J} + \frac{n-2}{n-1}\bar{H}^2\bar{J}; \quad \bar{J}' = \bar{H}\bar{J} \\ J'' \le -(n-1)k\,J + \frac{n-2}{n-1} H^2 J; \quad J' = HJ \\ J, \bar{J} \text{ has same "initial" data} \end{cases}$$

Let $f(r,\theta) = J^{\frac{1}{n-1}}(r,\theta)$ s.t. $\begin{cases} f' = \frac{1}{n-1} H f \\ f'' \le -\frac{1}{n-1} R_{rr} f \le -k f. \end{cases}$

$\left(\bar{f}(r) = \bar{J}^{\frac{1}{n-1}}(r), \quad \ldots \right.$

then $F \triangleq \dfrac{f(r,\theta)}{\bar{f}(r)}$ satisfies $(\bar{f}^2 F')' \leq 0$

using ODE of f and $\bar{f}$ wrt r.

Locally Euclidean (initial data)

$$\Rightarrow \begin{cases} f(\theta,0) = 0 \\ \bar{f}(0) = 0 \end{cases} \text{ and } \begin{cases} f'(\theta,0) = 1 \\ \bar{J}'(0) = 1 \end{cases}$$

$$\Rightarrow F' \leq 0 \quad \forall r > 0$$

$$\Rightarrow \underset{\frac{J'}{J}}{\underset{\parallel}{H(\theta,r)}} \leq \underset{\frac{\bar{J}'}{\bar{J}}}{\underset{\parallel}{\bar{H}(r)}} = \begin{cases} (n-1)\sqrt{k}\cot(\sqrt{k}r), & \text{if } k > 0 \\ (n-1)r^{-1} & \text{if } k = 0 \\ (n-1)\sqrt{-k}\coth(\sqrt{-k}r) & \text{if } k < 0. \end{cases}$$

<u>Remark</u>: This implies the Myer thm

because $\bar{H}\left(\frac{\pi}{\sqrt{k}}\right) = \infty \,/\, \bar{J}\left(\frac{\pi}{\sqrt{k}}\right) = 0$.

proof of volume comparison thm:

$$\boxed{\cdot \ \frac{V(p,r)}{V_k(r)} \text{ is non-increasing in } r>0.}$$

Recall: $C(r) = \left\{ v \in S_pM : \exp_p(sv) \text{ is minimizing up to } r \right\}$

clearly, $C(r_2) \subset C(r_1)$ if $r_2 > r_1$

Estimates above shows that $\forall\, \theta \in C(r_2)$,

$$\frac{J(\theta, r_1)}{\bar{J}(r_1)} \geq \frac{J(\theta, r_2)}{\bar{J}(r_2)} \quad \text{make sense} \quad \forall\, r_1 < r_2.$$

$$\Rightarrow \left(\int_{C(r_2)} J(\theta, r_1)\, d\theta\right) \cdot \bar{J}(r_2) \geq \left(\int_{C(r_2)} J(\theta, r_2)\, d\theta\right) \cdot \bar{J}(r_1)$$

$$\wedge\!|$$

$$\left(\int_{C(r_1)} J(\theta, r_1)\, d\theta\right) \cdot \bar{J}(r_2)$$

$$= A_p(r_1) \cdot \bar{J}(r_2)$$

$$\| \quad A_p(r_2) \cdot \bar{J}(r_1)$$

together with $\bar{A}(r) = \omega_{n-1} \cdot \bar{J}(r)$

$$\Rightarrow \quad \frac{A_p(r_2)}{\bar{A}(r_2)} \le \frac{A_p(r_1)}{\bar{A}(r_1)}, \quad \forall\, r_1 < r_2 .$$

Let
$$\begin{cases} F(r) = V_p(r) \\ G(r) = \bar{V}(r) \end{cases}$$
$\leftarrow$ volume on spaceform with $\bar{K}_g = k$.

we have
$$\frac{F'(r_2)}{G'(r_2)} \le \frac{F'(r_1)}{G'(r_1)} \quad \forall\, r_1 < r_2 .$$

(since by co-area formula, $\partial_r \mathrm{Vol}(B(r)) = \partial B(r)$)

" **Rmk**: F is lip-schitz in general!

so F' still makes sense. "

Baby choice: let $r_1 \to 0$

By locally Euclidean, $\Rightarrow \dfrac{F'(r)}{G'(r)} \le 1, \ \forall\, r > 0 .$

$$\Rightarrow \quad \mathrm{Vol}(B(p,r)) \le \mathrm{Vol}_k(\bar{B}(r)), \quad \forall r>0.$$

Known: $\quad \dfrac{F'(r)}{G'(r)} \le \dfrac{F'(s)}{G'(s)} \qquad \forall\, 0<s<r$

Consider $\quad \left(\dfrac{F}{G}\right)' = \dfrac{F'G - FG'}{G^2}$

$$\le \frac{\frac{F'(s)}{G'(s)}\, G'G - FG'}{G^2} \quad \text{at } r>s$$

At the same time,

$$\int_0^r G'(s)\,ds \le \int_0^r F'(s)\cdot \frac{G'(r)}{F'(r)}\,ds$$

$$G(r) \qquad\qquad F(r)\cdot \frac{G'(r)}{F'(r)}$$

$$\Rightarrow \left(\frac{F}{G}\right)'(r) \le \frac{1}{G^2}\left(\frac{F(s)}{G(s)}\cdot G'G - FG'\right), \quad s<r$$

Let $s \to r$

$$\Rightarrow \left(\frac{F}{G}\right)'(r) \leq \frac{1}{G^2}\left(\frac{F}{\cancel{G}} G' \cancel{G} - F G'\right)$$

$$\leq 0 \quad \# \quad (\text{at } r > 0).$$

$\therefore \quad \dfrac{V(p,r)}{\bar{V}(r)}$ is non-increasing in $r>0$.

$\#$

<u>Conseq</u>: ① If $Ric \geq 0$, then

$$V(p,r) \leq \omega_n r^n \quad \forall r>0, \forall p \in M.$$

② If in addition, $\exists p \in M, r>0$ s.t.

$$V(p,r) = \omega_n r^n.$$

then $B(p,r) \xrightarrow{\text{Isometric}} \bar{B}_{\mathbb{R}^n}(0,r)$

In particular, if (M,g) is complete mfd

with $Ric \geq 0$ and $\displaystyle\lim_{r\to\infty} \frac{V(p,r)}{\omega_n r^n} = 1$

then $(M, g) \cong (\mathbb{R}^n, \delta)$

<u>Sketch of pf</u>: If equality holds for some $r>0$,

then • $C(r) = C(s) \quad \forall\, r > s > 0$

$= S_pM$ (the whole sphere)

• $\bar{J}(s) = J(\theta, s) \quad \forall\, s < r$.

$\Rightarrow \quad A_{ij} = \frac{1}{n-1} H g_{ij}$ on each $\partial B(s)$.

Gauss lemma + ODE $\overset{\star}{\Rightarrow}$ $g = dr^2 + r^2 h_{S^{n-1}}$. #