Motivation:

under Normal coordinate at $p \in M$.

normal coord.

$$g_{ij} = \delta_{ij} + \cancel{\partial_k g_{ij}} \cdot x^k + \boxed{??} \, x^k x^l + o(r^2)$$

Curvature enters!!

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**Defn**: The curvature Rm of $(M, g)$ is a tri-linear map: $\Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$

given by $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

**Rmk**: If $(M,g) = (\mathbb{R}^n, \delta)$, then $R(X,Y)Z \equiv 0$

☆ Rm = measurement from Euclidean space.

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**Prop**: Rm is $(3,1)$ Tensor.

**pf**: suffices to check linear over $f \in C^\infty_{loc}(V)$. (local smooth fn)

$$R(fX, Y)Z = \nabla_{fX} \nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX, Y]} Z$$

$$= f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - f\nabla_{[X,Y]}Z$$

$$+ \nabla_{Y(f)X} Z$$

$$= f\left(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\right)$$

$$= f\,R(X,Y)Z \quad \#$$

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$\because$ $R(X,Y)Z = -R(Y,X)Z \quad \forall X, Y, Z$

$\therefore$ $R(X,fY)Z = f\,R(X,Y)Z.$

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$$R(X,Y)(fZ) = \nabla_X\nabla_Y(fZ) - \nabla_Y\nabla_X(fZ) - \nabla_{[X,Y]}(fZ)$$

$$= \nabla_X\left(Y(f)Z + f\nabla_Y Z\right)$$

$$- \nabla_Y\left(X(f)Z + f\nabla_X Z\right)$$

$$- [X,Y](f)\cdot Z + f\nabla_{[X,Y]}Z$$

$$= f\left[R(X,Y)Z\right] + \cancel{(XY)(f)}Z$$

$$+ \cancel{Y(f)\nabla_X Z} - \cancel{(YX)(f)}Z$$

$$- \cancel{X(f)}\nabla_Y Z + \cancel{X(f)}\nabla_Y Z$$

$$- \cancel{Y(f)}\nabla_X Z - \cancel{[X,Y](f)}\cdot Z$$

$$= f\,R(X,Y)Z \quad \#$$

$\therefore$ Rm is a (3,1) type tensor.

$$\boxed{\text{Locally, } Rm = R_{ijk}{}^{l}\, dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l}}$$

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In local coordinate,

$$R_{ijk}{}^{l}\,\partial_l = R(\partial_i, \partial_j)\partial_k$$

$$= \nabla_i \nabla_j \partial_k - \nabla_j \nabla_i \partial_k - \cancelto{0}{\nabla_{[\partial_i, \partial_j]}\partial_k}$$

$$= \nabla_i(\Gamma_{jk}^{p}\partial_p) - \nabla_j(\Gamma_{ik}^{p}\partial_p)$$

$$R_{ijk}{}^{l}\partial_l = \left(\partial_i \Gamma_{jk}^{\cancel{p}\,l} - \partial_j \Gamma_{ik}^{\cancel{p}\,l}\right)\cancel{\partial_p}\,\partial_l$$

$$+ \left(\Gamma_{jk}^{p}\Gamma_{ip}^{l} - \Gamma_{ik}^{p}\Gamma_{jp}^{l}\right)\partial_l$$

$$\Rightarrow \boxed{R_{ijk}{}^{l} = \partial_i \Gamma_{jk}^{l} - \partial_j \Gamma_{ik}^{l} + \Gamma_{jk}^{p}\Gamma_{ip}^{l} - \Gamma_{ik}^{p}\Gamma_{jp}^{l}}$$

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Example:

① $(\mathbb{R}^n, \delta)$, $R_{ijk}{}^{l} \equiv 0 \quad \forall i, j, k, l$

② $(S^n, g_{sph})$, $R_{ijk}{}^{l} = \left(\delta_i^l g_{jk} - g_{ik}\delta_j^l\right)$

③ $(\mathbb{H}^n, g_{hyp})$, $R_{ijk}{}^{l} = -\left(\delta_i^l g_{jk} - g_{ik}\delta_j^l\right)$

[ when $n=2$, $\frac{4\, dx\, dy}{(1-r^2)^2}$ ]

$$R(X,Y,Z,W) \triangleq \langle R(X,Y)Z, W\rangle$$

So that we may regard Rm as $(4,0)$ type tensor

Symmetry of Rm:

① (Bianchi identity)

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$$

② $R(X,Y,Z,W) = -R(Y,X,Z,W)$

$= -R(X,Y,W,Z)$

$= R(Z,W,X,Y)$

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pf of ①: Rm is a tensor.

$\therefore$ It suffices to verify using normal coordinates.

$$R_{ijkl} + R_{jkil} + R_{kijl}$$

$$= \langle \nabla_i \nabla_j \partial_k - \nabla_j \nabla_i \partial_k + \nabla_j \nabla_k \partial_i - \nabla_k \nabla_j \partial_i + \nabla_k \nabla_i \partial_j - \nabla_i \nabla_k \partial_j , \partial_l \rangle$$

$$\begin{aligned}
= \; & \partial_i \langle \nabla_j \partial_k , \partial_l \rangle - \langle \nabla_j \partial_k , \nabla_i \partial_l \rangle \\
& - \partial_j \langle \nabla_i \partial_k , \partial_l \rangle + \langle \nabla_i \partial_k , \nabla_j \partial_l \rangle \\
& + \partial_j \langle \nabla_k \partial_i , \partial_l \rangle - \langle \nabla_k \partial_i , \nabla_j \partial_l \rangle \\
& - \partial_k \langle \nabla_j \partial_i , \partial_l \rangle + \langle \nabla_j \partial_i , \nabla_k \partial_l \rangle \\
& + \partial_k \langle \nabla_i \partial_j , \partial_l \rangle - \langle \nabla_i \partial_j , \nabla_k \partial_l \rangle \\
& - \partial_i \langle \nabla_k \partial_j , \partial_l \rangle + \langle \nabla_k \partial_j , \nabla_i \partial_l \rangle
\end{aligned}$$

$$= 0 \quad \#$$

$(R_{ijkl} = -R_{ijlk})$: It suffices to show $\boxed{R_{ijXX} = 0 \quad \forall X.}$

Consider $X(s) = \partial_k + s\partial_l$, around $s=0$.

$\Rightarrow \partial_s\big|_{s=0} R_{ijXX} = 0 = R_{ijkl} + R_{ijlk}$ #.

LHS $= \langle \nabla_i\nabla_j X - \nabla_j\nabla_i X, X\rangle$

$= \partial_i\langle \nabla_j X, X\rangle - \langle \nabla_j X, \nabla_i X\rangle$

$- \partial_j\langle \nabla_i X, X\rangle + \langle \nabla_i X, \nabla_j X\rangle = 0$ #

Remains to prove $R_{ijkl} = R_{klij}$:

$$\begin{cases} R_{ijkl} + R_{jkil} + R_{kijl} = 0 \\ R_{jkli} + R_{klji} + R_{ljki} = 0 \\ R_{klij} + R_{lkij} + R_{likj} = 0 \\ R_{ijlk} + R_{jlik} + R_{lijk} = 0 \end{cases}$$

$\Rightarrow 0 = R_{kijl} + R_{iklj} + R_{ljki} + R_{jlik}$

$$0 = (R_{kijl} + R_{jlik}) \cdot 2.$$

Since $i, j, k, l$ are dummy,

we are done!! #.

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Q: Can we determine Rm by something simpler??

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Defn: Given a 2-dim plane $\sigma \subset T_pM$

then the sectional curvature $K(\sigma)$ of $\sigma$

is given by $R(u, v, v, u)$ where

$\{u, v\}$ are O.N. Basis of $\sigma$.

(Ex: Sectional curvature is indep of choice of $\{u, v\}$)

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Prop: The sectional curvature determines the curvature tensor.

(i.e. if $K(\sigma) = \tilde{K}(\sigma)$ $\forall$ plane $\sigma \subset T_pM$ then $Rm = \tilde{Rm}$)

pf:

Claim: $\nearrow$ prop Done.

$$6R(u,v,w,z) = \frac{d^2}{ds\,dt}\Big|_{\substack{s=0\\t=0}} \Big[ R(u+sz, v+tw, v+tw, u+sz) - R(u+sw, v+tz, v+tz, u+sw) \Big]$$

pf:

R.H.S = coeff. of $st$

$$= \boxed{R_{uvwz}} + \boxed{R_{zvwu} + R_{uwvz}} + \boxed{R_{zwvu}}$$

$$\boxed{- R_{uwzw}} \boxed{- R_{wvzu} - R_{uzwv}} \boxed{- R_{wzvu}}$$

$$\| \qquad\qquad\qquad\qquad \|$$
$$2R_{uvwz} \qquad\qquad\qquad \begin{array}{c} 2R_{zwvu} \\ \| \\ 2R_{uvwz} \end{array}$$

$$R_{zvwu} + R_{uwvz} - R_{wvzu} - R_{uzvw}$$

$$= 2\left(R_{zvwu} + R_{uzwv}\right) \overset{\text{(Bianchi)}}{=} 2R_{uvwz}$$

$$= 6R_{uvwz}. \quad \#$$

prop

$$\boxed{\begin{array}{l} \therefore \text{ If } K(\sigma) = \text{constant } c_0 \in \mathbb{R}. \\ \text{then } R_{ijkl} = c_0\left(g_{il}g_{jk} - g_{ik}g_{jl}\right) \end{array}}$$

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- $R_{ijkl}$ = curvature tensor

analogy

$D^2 f \leftrightarrow$ $K(\sigma)$ = sectional curvature of plane $\sigma$.

$\Delta f \leftrightarrow$ $R_{ij}$ = Ricci curvature

if $\{e_i\}$ o.N.

$$Ric(X,Y) = g^{ij} R_{XijY} = \sum_{i=1}^{n} R_{XiiY}$$

$\hat{\Delta f} \leftrightarrow R$ = scalar curvature $\left(\in C^\infty(M)\right)$

$$R = \mathrm{tr}_g R_{ii} = g^{ij} R_{ij} \overset{\text{if } \{e_i\} = ON}{=} \sum_{i=1}^{n} R_{ii}$$

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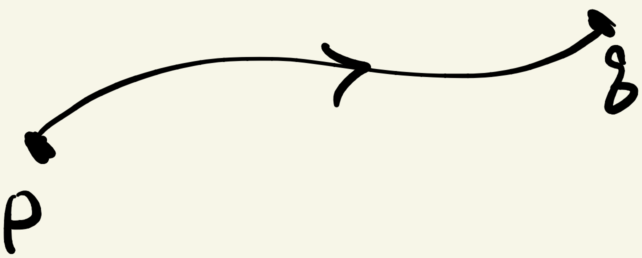



variation parameter

Consider $\alpha : [a,b] \times (-\varepsilon, \varepsilon) \times (-\delta, \delta) \to M$

s.t. $\alpha(t, 0, 0) = \gamma(t)$ = normal geod. from $p$ to $q$.

Study $L(\alpha)$ = Length of $\alpha(t, v, w)$.

$$\frac{\partial L}{\partial v} = \int_a^b \frac{1}{|T|} \langle \nabla_T V, T \rangle \, dt.$$

$$\left.\frac{\partial^2 L}{\partial w \partial v}\right|_{(0,0)} = \frac{\partial}{\partial w} \left( \int_a^b \frac{1}{|T|} \langle \nabla_T V, T \rangle \, dt \right)$$

$$= \int_a^b \Big[ \frac{1}{|T|} \cdot \langle \nabla_W \nabla_T V, T \rangle$$

$$+ \frac{1}{|T|} \cdot \langle \nabla_T V, \nabla_W T \rangle$$

$$- \frac{1}{|T|^3} \langle \nabla_T V, T \rangle \langle \nabla_W T, T \rangle \Big] dt$$

$|T| = 1$

$$= \int_a^b T \langle \nabla_T \nabla_W V, T \rangle + R(W, T, V, T)$$

$$+ \langle \nabla_T V, \nabla_W T \rangle - \langle \nabla_T V, T \rangle \langle \nabla_W T, T \rangle$$

$$= \underbrace{\langle \nabla_W V, T \rangle \Big|_a^b}_{①} - \int_a^b \underbrace{\langle \nabla_W T, T \rangle \langle \nabla_W T, T \rangle}_{②}$$

$$+ \int_a^b \underbrace{\langle \nabla_T V, \nabla_T W \rangle}_{③} - R(W, T, T, V)$$

Ⓐ If $\begin{cases} \alpha(a, u, v) = p \\ \alpha(b, u, v) = q. \end{cases}$ $\forall u \in (-\varepsilon, \varepsilon)$, $v \in (-\delta, \delta)$

then ① $\equiv 0$

Ⓑ : ② $= 0$ since $\begin{cases} \langle \nabla_W T, T \rangle = \frac{1}{2} V(|T|^2) = 0 \\ \langle \nabla_W T, T \rangle = 0 \end{cases}$

(c) If $V, W$ are parallel wrt $\gamma$

then ③ $\equiv 0$.

Then Rm determine the 2nd variation of arc-length

**Thm** (Meyer) Suppose $(M,g)$ is a complete Riemannian mfd with $Ric(X,X) \geq (n-1)$ $\forall$ $|X|=1$. (compare with sphere)

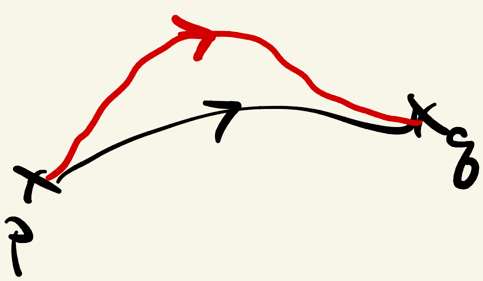
then $M$ must be cpt. Moreover

$diam(M,g) \leq \pi$.

pf: Suffices to show that $d(p,q) \leq \pi$ $\forall$ $p,q \in M$.

Let $p,q \in M$ s.t. $d(p,q) = \ell > 0$ and Realized by a minimal normal geodesic $\gamma : [0,\ell] \to M$.

along $\gamma$




let $\{E_i(t)\}_{i=1}^{n}$ be a parallel O.N. frame along $\gamma(t)$.

where $E_n = \gamma'(t)$

$$\boxed{dt\left(\frac{\partial}{\partial W}\right) = W = V_i}$$

Consider $\alpha_i(t,s) = \exp_{\gamma(t)}(s\,V_i(t))$ where

$$V_i(t) = \sin\left(\frac{\pi t}{l}\right) E_i(t), \quad \text{for } i = 1, 2, \dots, n-1$$

s.t. $\begin{cases} \alpha_i(0,s) = p \\ \alpha_i(l,s) = q \end{cases}$

$$0 \in \sum_{i=1}^{n} \frac{\partial^2 L(\alpha_i)}{\partial s^2}\Big|_{s=0} = \sum_{i=1}^{n} \int_0^l |\nabla_T V_i|^2 - R(V_i, T, T, V_i)\, dt$$

$$\sum_{i=1}^{n} \sin^2\left(\frac{\pi t}{l}\right) \cdot R(E_i, E_n, E_n, E_i) = \sin^2\left(\frac{\pi t}{l}\right) \cdot R_{ic}(\gamma', \gamma') \geq \sin^2\left(\frac{\pi t}{l}\right) \cdot (n-1)$$

$E_i$ parallel wrt $\gamma'$

$$\nabla_T V_i = \nabla_T\left(\sin\left(\frac{\pi t}{l}\right) \cdot E_i\right) = \left[\frac{\partial}{\partial t} \sin\left(\frac{\pi t}{l}\right)\right] E_i$$

$$= \frac{\pi}{l} \cos\left(\frac{\pi t}{l}\right) \cdot E_i(t).$$

$$\Rightarrow \sum_{i=1}^{n-1} |\nabla_T V_i|^2 = \left(\frac{\pi}{l}\right)^2 \cos^2\left(\frac{\pi t}{l}\right) \cdot (n-1).$$

$$\Rightarrow 0 \le \left( \sum_{i=1}^{n-1} \frac{d^2}{ds^2}\Big|_{s=0} L(\alpha_i) \right) \left(\frac{1}{n-1}\right)$$

minimal geodesic

$$\le \int_0^{\ell} \left(\frac{\pi}{\ell}\right)^2 \cos^2\left(\frac{\pi t}{\ell}\right) - \sin^2\left(\frac{\pi t}{\ell}\right) dt.$$

$$= \frac{1}{2}\left(\frac{\pi}{\ell}\right)^2 \ell - \frac{1}{2}\ell$$

$$\Rightarrow \quad \ell \le \pi \quad \#.$$

$$\therefore \quad d(p,q) \le \pi \quad \forall\, p, q \in M. \quad \#.$$

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Consider the variation as follows:

$$\alpha(t,s) = \exp_p\left(t(v+sw)\right), \quad v, w \in T_pM.$$

$$\alpha(t,0) = \exp_p(tv) = \text{geo.}$$

$W(0) = 0$

then what does $W = d\alpha(\partial s)$ tell us ??

$$\nabla_T \nabla_T W = \nabla_T \nabla_W T = \nabla_W \underbrace{\nabla_T T}_{0 \text{ (geodesic)}} + R(T,W)T$$

Satisfies ODE:

$$\nabla_T \nabla_T W + R(W,T)T = 0 \quad \text{(ODE along } \gamma)$$

**Defn.**

If $W$ is a V.F along $\gamma : [a,b] \to M$ s.t.

$$\nabla_{\gamma'} \nabla_{\gamma'} W + R(W, \gamma')\gamma' = 0 \quad \forall t \in [a,b]$$

Then $W$ is called the $\boxed{\text{Jacobi field}}$ along $\gamma$.

**Rmk**: J-field is a 2nd order ODE determined by $J(0)$, $J'(0)$: to see this

take $\{E_i(t)\}$ parallel o.n. along $\gamma$.

$J = \langle J, E_i \rangle E_i$ satisfy J.F. eqn iff

$$\nabla_{\gamma'} \nabla_{\gamma'} J + R(J, \gamma')\gamma' = 0$$

$\Leftrightarrow$ $J_i = \langle J, E_i \rangle$ satisfies

$$\left.\begin{aligned} J_i'' &= \langle \nabla_{\gamma'} \nabla_{\gamma'} J, E_i \rangle \\ &= -R(J, \gamma', \gamma', E_i) \end{aligned}\right\} \text{2nd. linear ODE.}$$

$= - J_j R_{j \cdot \cdot i} \Big\}$

fcn along $\gamma(t)$.

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Example: when $K(\gamma) = K$.

J field (w/ $J(0)=0$) has sol in form of

$$J(t) = \begin{cases} \dfrac{\sin(t\sqrt{K})}{\sqrt{K}} w(t) & \text{if } K>0 \\ t\, w(t) & \text{if } K=0 \\ \dfrac{\sinh(t\sqrt{-K})}{\sqrt{-K}} w(t) & \text{if } K<0 \end{cases}$$

where $w \perp \gamma'$, $\nabla_{\gamma'} w = 0$ and $|w| = 1$.

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prop: Given a J.F. along $\gamma(t) = \exp_p(tv)$

$J(0)$ and $J'(0) = w \in T_pM$, then

$$J(t) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_p(tv(s))$$

where $v(0) = v = \gamma'(0)$ and $v'(0) = w$.

pf: define $\bar{J}(t) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_p(tv(s))$

$$\begin{cases} \bar{J}(0) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_p(0) = 0. \\ \frac{d}{dt}\Big|_{t=0} \bar{J}(t) = \frac{d}{dt}\Big|_{t=0} \left[ d\exp_p\big|_{tv}(tw) \right] \\ \qquad = d\exp_p\big|_{tv}(w)\Big|_{t=0} = w \end{cases}$$

$\Rightarrow \quad J \equiv \bar{J}$ by uniqueness of ODE.