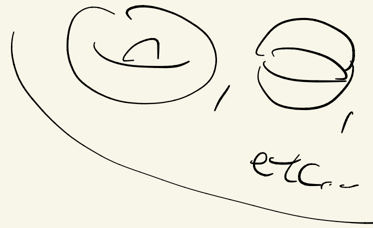
Last time: Ric lower bdd $\rightsquigarrow$ comparison geometry (volume, Laplacian comparison)

Last time: Bochner technique $\rightsquigarrow$ what else geometric properties you can conclude??

In PDE. Let M be a compact mfd w/o boundary



$$\boxed{\begin{array}{l}\text{Poincaré inequality: } \exists\, C_M>0 \text{ s.t. } \forall u \in W^{1,2}(M) \quad (\forall u \in C^\infty(M)) \\ \displaystyle\int_M |u-\bar{u}|^2 \, d\mu \le C_M \int_M |\nabla u|^2 \, d\mu \\ \text{where } \bar{u} = ⨍_M u \, d\mu .\end{array}}$$

Rmk: • If we are ~~Are~~ Not asking for optimal constant C_M, then we can prove using covering argument with Euclidean Poincaré ineq.

Alternatively, may prove by contradiction argument with weak compactness.

• the optimal Poincaré constant depends on the geometry.

Q: Can we estimate best Poincaré inequality constant??

Naively, consider
$$m = \inf\left\{ \frac{\int_M |\nabla u|^2 \, d\mu}{\int_M u^2 \, d\mu} \;\middle|\; \begin{array}{l} u \in C^\infty \\ \bar{u} = 0,\ u \not\equiv 0 \end{array} \right\} \ge 0$$

Question: Can we find the minimizer u realising m??

Formally: If u is a minimizer, then $\forall\, v \in W^{1,2}$ s.t. $\int_M v = 0$, we have $u(t) = u + tv \in Y = \{ f \in W^{1,2} : \bar{f} = 0 \}$, $\forall\, t \in \mathbb{R}$.

and $f(t) = \dfrac{\int_M |\nabla u(t)|^2}{\int_M |u(t)|^2}$ satisfies $\begin{cases} ① & f(0) = m. \\ ② & f(t) \geq m, \forall t. \end{cases}$

$$\Rightarrow f'(0) = 0 \quad \forall \text{ chosen } v \in Y.$$

$$f'(0) = \frac{\left(\int_M u^2\right)\left(2\int_M 2\langle \nabla u, \nabla v\rangle\right) - \left(2\int_M uv\right)\left(\int_M |\nabla u|^2\right)}{\left(\int_M u^2\right)^2} = 0$$

$$\Rightarrow \left(\int_M u \cdot v\right)\left(\int_M |\nabla u|^2\right) = \left(\int_M u^2\right)\left(\int_M \langle \nabla u . \nabla v\rangle\right)$$

$$\| $$

$$m \left(\int_M u \cdot v\right)\left(\int u^2\right)$$

$$\boxed{\Rightarrow \int_M \left(\langle \nabla u, \nabla v\rangle - m u v\right) d\mu = 0, \ \forall v \in W^{1,2} \text{ s.t. } \bar{v} = 0}$$

$\|$ (stoke thm)

$$\int_M -v\,(mu + \Delta u)\, d\mu.$$

$\forall\, v \in W^{1,2}$, $\tilde{v} = v - \bar{v}$ s.t. $\int_M \tilde{v}\, d\mu = 0$.

$\therefore\ \tilde{v} \in Y.$

$$\Rightarrow 0 = \int_M \tilde{v}\,(mu + \Delta u) = \int_M v\,(mu + \Delta u) - \bar{v} \int_M \cancel{mu} + \cancel{\Delta u}$$

$\bar{v} \in \mathbb{R}$, $\quad (\int u = 0)$

$$\Rightarrow \int_M (mu + \Delta u)\, v = 0 \qquad \forall v \in W^{1,2}(M)$$

(Formally)

$\Rightarrow$ u satisfies
$$\begin{cases} \Delta u = -mu, & m \geq 0 \\ \int_M u = 0, & u \not\equiv 0. \end{cases}$$

If $m = 0$, this is impossible because u in this case is harmonic. Simple maximum principle $\Rightarrow$ $u \equiv$ constant

$$\Rightarrow u \equiv 0 \quad \left(\int_M u = 0\right).$$

If $m > 0$, $\int_M -mu = \int_M \Delta u = 0$ (stoke thm.)

Hence, to estimate the Poincare constant, it suffices to study

$$* \begin{cases} \Delta u = -\lambda u, \ \lambda > 0 \\ u \not\equiv 0 \end{cases} \quad \text{on } M.$$

$\boxed{\text{First eigenvalue}}$

Thm (Lichnerowicz and Obata)

Let M be compact mfd w/o boundary with $Ri \geq (n-1)k$, $k > 0$, then $\lambda_1 \geq nk$.

Moreover, $\lambda_1 = nk$ iff $M \cong S^n_{\frac{1}{\sqrt{k}}}$..

pf: By scaling, may assume $R = 1$ $(g \mapsto kg)$

Let $\begin{cases} \Delta u = -\lambda_1 u & , \lambda > 0 \\ u \not\equiv 0. \end{cases}$

$$\begin{cases} \Delta|\nabla u|^2 = 2|\nabla^2 u|^2 + 2\,\mathrm{Ric}(\nabla u, \nabla u) + 2\langle \nabla u, \nabla \Delta u\rangle \\ \qquad \geq 2|\nabla^2 u|^2 + 2(n-1)|\nabla u|^2 - 2\lambda_1 |\nabla u|^2. \\ \Delta u^2 = 2u\Delta u + 2|\nabla u|^2 = -2\lambda_1 u^2 + 2|\nabla u|^2 \end{cases}$$

Improved Kato inequality: $|\nabla^2 u|^2 = \sum_{i,j} u_{ij}^2$

$$\geq \sum_{i=1}^n u_{ii}^2 \overset{\text{(cauchy)}}{\geq} \Big(\sum_{i=1}^n u_{ii}\Big)^2 \Big(\frac{1}{n}\Big)$$

$$= \frac{1}{n}(\Delta u)^2 = \frac{1}{n}\lambda_1^2 u^2. \quad \text{Good}$$

Let $\Lambda > 0$,

$$\Delta\big(\underbrace{|\nabla u|^2 + \Lambda u^2}_{F}\big) \geq 2\lambda_1 u^2\Big(\frac{1}{n}\lambda_1 - \Lambda\Big)$$

$$+ 2|\nabla u|^2\big((n-1) - \lambda_1 + \Lambda\big)$$

We choose $\Lambda = \frac{\lambda_1}{n}$ s.t. $\Delta F \geq 2|\nabla u|^2\Big((n-1) - \lambda_1 + \frac{\lambda_1}{n}\Big)$

$$= 2|\nabla u|^2\Big((n-1) + \frac{\lambda_1}{n}(1-n)\Big)$$

$$= 2(n-1)|\nabla u|^2\Big(1 - \frac{\lambda_1}{n}\Big)$$

$$\Rightarrow \int_M \Delta F \, d\mu = 0 \geq 2(n-1)\left(1-\frac{\lambda_1}{n}\right) \underbrace{\int_M |\nabla u|^2}_{\neq 0}$$

$\Rightarrow \lambda_1 \geq n$. (this proved lower bdd).

Moreover, if $\lambda_1 = n$. $\Rightarrow \Delta F \geq 0$

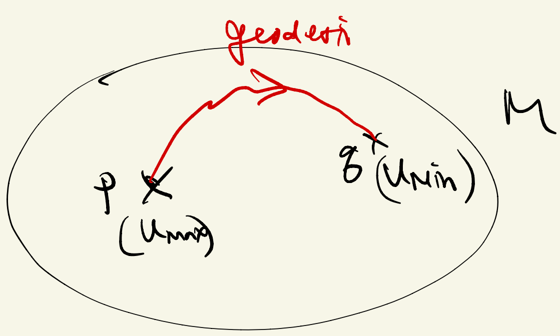
$\therefore$ Max principle implies $F \equiv$ constant. ($M =$ cpt)

$$* \quad |\nabla u|^2 + u^2 = \text{const.} = u_\infty^2 \left(= \|u\|_\infty^2\right)$$

(Recall $\begin{cases} \Delta u = -\lambda_1 u \\ u \not\equiv 0 \end{cases}$, we may normalize s.t. $\sup_M u = 1$)

Evaluate $*$ at $\min_M u$, $u_{\min}^2 = 1 \Rightarrow u_{\min} = -1$

$$\therefore \quad \frac{|\nabla u|}{\sqrt{1-u^2}} = 1. \quad \text{on } M.$$



$$\begin{cases} u(p) = u_{\max} = 1 \\ u(q) = u_{\min} = -1. \end{cases}$$

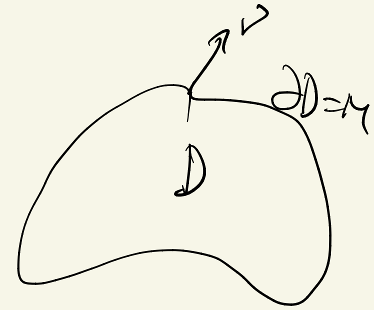
$$\Rightarrow \quad \operatorname{diam}(M) \geq \int_\gamma ds = \int_\gamma \frac{|\nabla u|}{\sqrt{1-u^2}} ds \overset{\text{co-area}}{=} \int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} = \pi.$$

($\gamma : [a,b] \to M$ is normal geod. s.t. $\gamma(a) = p$, $\gamma(b) = q$)

But by volume comparison, $\operatorname{diam}(M) \leq \pi$

$$\Rightarrow \quad \operatorname{diam}(M) = \pi \quad \Rightarrow \quad M \cong S^n \quad \text{(Cheng)}.$$

More applications of Bochner formula:



Let (D^{n+1}, g) be a cpt mfd w/ $\partial D = M^n$

think of $M \subset D$ (as a sub-mfd)

$\rightsquigarrow$ talk about the 2nd fundamental form, mean curvature, etc.

Notations: on D^{n+1}, $\bar{\nabla}$, $\bar{\Delta}$

on M^n, ∇, Δ $\qquad A_{ij} = -\langle \bar{\nabla}_i \partial_j, \nu \rangle$, $H = g^{ij} A_{ij}$.

Reilly formula: $\forall f \in C^\infty(D)$, we have

$$\frac{n}{n+1} \int_D (\bar{\Delta} f)^2 \geq \int_D \overline{Ric}(\bar{\nabla} f, \bar{\nabla} f)$$

$$+ \int_M H f_\nu^2 + \underbrace{2 (\Delta f)(f_\nu) + A(\nabla f, \nabla f)}_{\text{Boundary term}}.$$

By Bochner formula!

pf: on D, $\frac{1}{2} \bar{\Delta} |\bar{\nabla} f|^2 = |\bar{\nabla}^2 f|^2 + \overline{Ric}(\bar{\nabla} f, \bar{\nabla} f) + \langle \bar{\nabla} f, \bar{\nabla} \bar{\Delta} f \rangle$

LHS: $\displaystyle \int_D \frac{1}{2} \bar{\Delta} |\bar{\nabla} f|^2 = \frac{1}{2} \int_M \nu (|\bar{\nabla} f|^2)$

on M, $\displaystyle \nu |\bar{\nabla} f|^2 = \partial_\nu (f_i^2) = \sum_{i=1}^{n+1} 2 f_i f_{i\nu}$

$$\overset{\triangle}{=} 2 f_\nu f_{\nu\nu} + 2 \sum_{i=1}^{n} f_{i\nu} f_i \qquad \left(\{\partial_i\}_{i=1}^{n} : \text{coord of } M \right)$$

$\left(f_{ij} = \overline{\nabla}_i \overline{\nabla}_j f_\cdot \right)$

$$= 2 f_\nu \left(\overline{\Delta} f - \sum_{i=1}^n f_{ii} \right) + 2 \sum_{i=1}^n f_{\nu i} f_i$$

$$= 2 f_\nu \left(\overline{\Delta} f - \Delta f - \sum_{i=1}^n A_{ii} f_\nu \right) + 2 \sum_{i=1}^n f_{\nu i} f_i$$

$$\left(f_{\nu i} = \overline{\nabla}_i \overline{\nabla}_\nu f = \partial_i (\partial_\nu f) - (\overline{\nabla}_i \nu) f \right.$$
$$\left. = \partial_i f_\nu - A_i^j f_j \right)$$

$$= 2 f_\nu \left(\overline{\Delta} f - \Delta f - H f_\nu \right) + 2 \sum_{i=1}^n f_i \left(\nabla_i f_\nu - A_{ij} f_j \right)$$

$$\therefore \frac{1}{2} \int_D \overline{\Delta} |\overline{\nabla} f|^2 = \int_M f_\nu \left(\overline{\Delta} f - \Delta f - H f_\nu \right)$$

$$+ \int_M \langle \nabla f, \nabla f_\nu \rangle - A(\nabla f, \nabla f)$$

(stoke thm)

$$= \int_M f_\nu \cdot \overline{\Delta} f - H f_\nu^2 - A(\nabla f, \nabla f) - 2 (\Delta f) f_\nu$$

#

R.H.S :

$$\int_D |\overline{\nabla}^2 f|^2 + \overline{Ric}(\overline{\nabla} f, \overline{\nabla} f) + \langle \overline{\nabla} f, \overline{\nabla} \overline{\Delta} f \rangle$$

stoke thm.

$$\geq \frac{1}{n+1} \int_D (\overline{\Delta} f)^2 + \int_D \overline{Ric}(\overline{\nabla} f, \overline{\nabla} f) + \int_M \overline{\Delta} f \cdot f_\nu - \int_D (\overline{\Delta} f)^2 .$$

Re-arrange : Reilly formula : $\forall f \in C^\infty(D)$.

$$\frac{n}{n+1} \int_D (\overline{\Delta} f)^2 \geq \int_D \overline{Ric}(\overline{\nabla} f, \overline{\nabla} f) + \int_M H f_\nu^2 + 2 (\Delta f) f_\nu + A(\nabla f, \nabla f)$$

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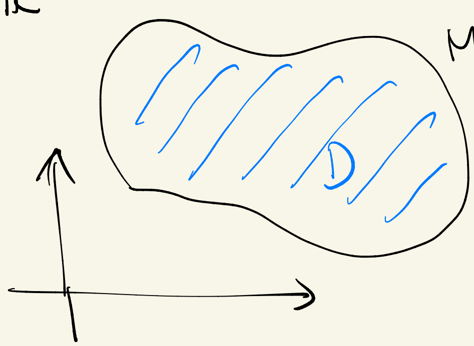
Thm (Alexandrov) Any cpt embedded hypersurface of positive constant mean curvature in $\mathbb{R}^{n+1}$ is sphere.

pf: By using Poisson kernel (some PDE theory)

$$\exists f \in C^\infty(D) \quad \text{s.t.} \quad \begin{cases} \bar{\Delta} f = -1 & \text{on } D \\ f = 0 & \text{on } \partial D = M \end{cases}$$

where M is the cpt embedded hypersurface with $H = \text{const} > 0$

$\mathbb{R}^{n+1}$



$M^n = \partial D$ may assume $H = n$ on M by scaling.

Reilly $\Rightarrow$ $$\frac{n}{n+1}\int_D (-1)^2 \geq \int_M n \cdot f_\nu^2 .$$

$$\Rightarrow \quad |D| \geq (n+1) \int_M f_\nu^2$$

Cauchy $\Rightarrow$ $$|D|^2 = \left(\int_D \bar{\Delta} f\right)^2 = \left(\int_M f_\nu\right)^2$$

$$\leq \left(\int_M f_\nu^2\right)\left(\int_M 1\right) = |M| \cdot \left(\int_M f_\nu^2\right)$$

$$\leq \frac{|M|}{n+1} \cdot |D| .$$

$$\Rightarrow \quad (n+1)\, |D| \leq |M| .$$

Claim: Equality holds

pf: Let $\vec{X}$ be a position vector from some points

$$\Delta_M \vec{X} \overset{(Ex)}{=} \vec{H} = \pm n \nu$$

$$\Rightarrow \frac{1}{2}\Delta_M |\vec{X}|^2 = \langle \Delta \vec{X}, \vec{X} \rangle + \langle \nabla \vec{X}, \nabla \vec{X} \rangle$$

$$= \langle \pm n \vec{\nu}, \vec{X} \rangle + n.$$

$$\Rightarrow |M| = \int_M 1 = \int_M \langle \vec{x}, \vec{\nu} \rangle = \int_D \operatorname{div} \vec{X} = (n+1)|D|.$$

Equality case holds $\Rightarrow$ $\begin{cases} f_\nu = \text{constant.} \\ \text{Equality case in Bochner formula.} \end{cases}$

$\Downarrow$

on D, $f_{ij} = \dfrac{-\delta_{ij}}{n+1}$

If we fix p s.t. $f(p) = \max$, then by translation,

we may assume $p = \text{origin} \in O$.

And hence $f(x) = -\dfrac{|x|^2}{2(n+1)} + \text{const.}$ on D

And $M = \{ f = 0 \} = \{ |x| = 2(n+1) \cdot \text{const.} \}$

$\Vert$

sphere.

End of Course!!