Suggested Solution to Assignment 5

Exercise 5.1

2. (a)

$$A_m = 2\int_0^1 x^2 \sin m\pi x\ dx = -2\frac{x^2}{m\pi}\cos m\pi x\Big|_0^1 + \int_0^1 \frac{4x}{m\pi}\cos m\pi x\ dx$$
$$= \frac{2(-1)^{m+1}}{m\pi} + \frac{4(-1)^m - 4}{m^3\pi^3}.$$

(b)

$$A_m = 2\int_0^1 x^2 \cos m\pi x dx = 2\frac{x^2}{m\pi}\sin m\pi x\Big|_0^1 - \int_0^1 \frac{4x}{m\pi}\sin m\pi x dx = (-1)^m\frac{4}{m^2\pi^2}. \qquad \square$$

4. To find the Fourier series of the function $f(x) = |\sin x|$, we first note that this is an even function so that it has a cos-series. If we integrate from 0 to π and multiply the result by 2, we can take the function $\sin x$ instead of $|\sin x|$ so that

$$a_0 = \frac{2}{\pi}\int_0^\pi \sin x dx = \frac{4}{\pi}.$$

$$a_n = \frac{2}{\pi}\int_0^\pi \sin x\cos nx dx = \begin{cases} \frac{4}{(1-n^2)\pi} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}.$$

Hence, we have

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \cdots\right).$$

Substituting $x = 0$ and $x = \frac{\pi}{2}$, we have

$$\sum_{n=1}^\infty \frac{1}{4n^2-1} = \frac{1}{2}.$$

$$\sum_{n=1}^\infty \frac{(-1)^n}{4n^2-1} = \frac{1}{2} - \frac{\pi}{4}. \qquad \square$$

5. (a) From Page.109, we have

$$x = \sum_{m=1}^\infty (-1)^{m+1}\frac{2l}{m\pi}\sin\frac{m\pi x}{l}.$$

Integration of both sides gives

$$\frac{x^2}{2} = c + \sum_{m=1}^\infty (-1)^m\frac{2l^2}{m^2\pi^2}\cos\frac{m\pi x}{l}.$$

The constant of the integration is the missing coefficient

$$c = \frac{A_0}{2} = \frac{1}{l}\int_0^l \frac{x^2}{2}dx = \frac{l^2}{6}.$$

(b) By setting $x = 0$ gives

$$0 = \frac{l^2}{6} + \sum_{m=1}^\infty (-1)^m\frac{2l^2}{m^2\pi^2},$$

or

$$\frac{\pi^2}{12} = \sum_{m=1}^\infty \frac{(-1)^{m+1}}{m^2}. \qquad \square$$

8. The key point in the problem above is to solve the following PDE problem.

$$u_t - u_{xx} = 0, \quad u(x,0) = \phi(x), \quad u(0,t) = u(l,t) = 0,$$

$$\phi(x) = \begin{cases} \frac{3}{2}, & 0 < x < \frac{2}{3}, \\ 3 - 3x, & \frac{2}{3} < x < 1 \end{cases}.$$

Through a standard procedure of separation variable method, we obtain

$$u(x,t) = \sum a_n e^{-n^2\pi^2 t} \sin n\pi x,$$

where $a_n = 2\int_0^1 \phi(x) \sin n\pi x dx = \frac{9}{n^2\pi^2} \sin \frac{2\pi n}{3}$, so the solution $T = u(x,t) + x$. □

9. From Section 4.2.7, we see that the general formula to wave equation with Neu- mann boundary condition is

$$u(x,t) = \frac{1}{2}(A_0 + B_0 t) + \sum_{n=1}^{\infty}(A_n \cos nct + B_n \sin nct) \cos nx,$$

where

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nx, \quad \psi(x) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} ncB_n \cos nx.$$

By further calculation, we have $B_0 = 1$, $B_2 = \frac{1}{4c}$ and the other coefficients are all zero. Hence, the solution is

$$u(x,t) = \frac{1}{2}t + \frac{\sin 2ct \cos 2x}{4c}. \qquad \square$$

Exercise 5.2

1. (a)Odd, period= $2\pi/a$;

   (b)neither even nor odd nor periodic;

   (c)even if m is even, odd if m is odd, and not periodic;

   (d)even, not periodic;

   (e)even, period= $b\pi$;

   (f)odd, not periodic.

2. Suppose $\alpha = p/q$, where p, q are co-prime to each other. Then is is not difficult to see that $S = 2q\pi$ is a period of the function. Suppose $2q\pi = mT$, where T is the minimal period. Then

$$\cos x + \cos \alpha x = \cos(x + T) + \cos(\alpha x + \alpha T).$$

Let $x = 0$, we have the above equality holds iff q/m, p/m are both integers. Therefore, $m = 1$. Hence, we finish the problem. □

4. $\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n cos\frac{n\pi x}{l} + B_n sin\frac{n\pi x}{l})$ where $A_n = \frac{1}{l}\int_{-l}^{l}\phi(x)cos\frac{n\pi x}{l}dx(n = 0, 1, 2, ...)$ and $B_n = \frac{1}{l}\int_{-l}^{l}\phi(x)sin\frac{n\pi x}{l}dx(n = 1, 2, ...)$.

   (a)$\phi(x)$ is an odd function and $cos\frac{n\pi x}{l}$ is an even function, thus, by (5), $A_n = 0$.

   (b)$\phi(x)$ is an even function and $sin\frac{n\pi x}{l}$ is an odd function, thus, by (5), $B_n = 0$.

5. Let $a_m = \frac{2}{l}\int_0^l \phi(x) \sin \frac{m\pi x}{l}$. Then we have

$$\phi(x) = \sum_{m=1}^{\infty} a_m \sin \frac{m\pi x}{l}. \qquad \square$$

7. The full series on $(-\pi,\pi)$ is $\widetilde{\phi}(x') = \frac{1}{2}A_0+\sum_{n=1}^{\infty}(A_n cosnx'+B_n sinnx'$ where $A_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\widetilde{\phi}(x)cosnxdx(n = 0,1,2,...)$ and $B_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\widetilde{\phi}(x)sinnxdx(n = 1,2,...)$. Set $x' = (\pi/l)x$ we obtain $\phi(x) = \widetilde{\phi}((\pi/l)x) = \frac{1}{2}A_0+\sum_{n=1}^{\infty}(A_n cos\frac{n\pi x}{l} + B_n sin\frac{n\pi x}{l})$ where $A_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\widetilde{\phi}(x)cosnxdx = \frac{1}{\pi}\int_{-l}^{l}\widetilde{\phi}((\pi/l)x)cos(n(\pi/l)x)d((\pi/l)x) = \frac{1}{l}\int_{-l}^{l}\phi(x)cos\frac{n\pi x}{l}dx(n = 0,1,2,...)$ and $B_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\widetilde{\phi}(x)sinnxdx = \frac{1}{\pi}\int_{-l}^{l}\widetilde{\phi}((\pi/l)x)sin(n(\pi/l)x)d((\pi/l)x) = \frac{1}{l}\int_{-l}^{l}\phi(x)sin\frac{n\pi x}{l}dx(n = 1,2,...)$.

9. $a_n = \frac{1}{\pi}(\int_0^{\pi}\phi(x)sinnxdx + \int_{-\pi}^{0}\phi(x)sinnxdx)= \frac{1}{\pi}(\int_0^{\pi}\phi(x)sinnxdx + \int_0^{\pi}\phi(x-\pi)sin(n(x-\pi))d(x-\pi))= \frac{1}{\pi}(\int_0^{\pi}\phi(x)sinnxdx + \int_0^{\pi}(-1)^n\phi(x)sinnxdx)= 0$ if n is odd.

10. (a) If ϕ is continuos on $(0,l)$, ϕ_{odd} is continuous on $(-l,l)$ if and only if $\lim\limits_{x\to 0^+}\phi(x) = 0$.

(b) If $\phi(x)$ is differentiable on $(0,l)$, ϕ_{odd} is differentiable on $(-l,l)$ if and only if $\lim\limits_{x\to 0^+}\phi'(x)$ exists, since ϕ'_{odd} is an even function, so the only thing to avoid is an infinite discontinuity at $x = 0$.

(c) If ϕ is continuos on $(0,l)$, ϕ_{even} is continuous on $(-l,l)$ if and only if $\lim\limits_{x\to 0^+}\phi(x)$ exists, since the only thing to avoid is an infinite discontinuity at $x = 0$.

(d) If $\phi(x)$ is differentiable on $(0,l)$, ϕ_{even} is differentiable on $(-l,l)$ if and only if $\lim\limits_{x\to 0^+}\phi'(x) = 0$, since ϕ'_{even} is an odd function. □

Extra. $u(0,t) = u(1,t) = 0$ tells us we can do odd extension and periodic extension with period 2. Thus define

$$\phi(x) = \begin{cases} \sin^2(\pi x), & x \in [2n, 2n+1] \\ -\sin^2(\pi x), & x \in [2n-1, 2n] \end{cases}$$

$$\psi(x) = \begin{cases} x(1-x), & x \in [2n, 2n+1] \\ x(1+x), & x \in [2n-1, 2n] \end{cases}$$

$n = 0, \pm 1, \pm 2, ...$ By d'Alembert's formula,$u(x,t) = \frac{1}{2}[\phi(x+2t)+\phi(x-2t)] + \frac{1}{4}\int_{x-2t}^{x+2t}\psi(s)ds$ solves the problem.

Exercise 5.3

3. Since $X(0) = 0$, by the odd extension $x(-x) = -X(x)$ for $-l < x < 0$, then X satisfies $X'' + \lambda X = 0$, $X'(-l) = X'(l) = 0$. Hence,

$$\lambda = [(n+\frac{1}{2})\pi]^2/l^2,\ X_n(x) = \sin[(n+\frac{1}{2})\pi x/l],\ n = 0,1,2,\dots$$

Thus we botain the general formula to this equation

$$u(x,t) = \sum_{n=0}^{\infty}[A_n\cos\frac{(n+\frac{1}{2})\pi ct}{l} + B_n\sin\frac{(n+\frac{1}{2})\pi ct}{l}]\sin\frac{(n+\frac{1}{2})\pi x}{l}.$$

By the boundry condition, we obtained that B_n are all zero, while $A_n = \frac{2}{l}\int_0^l \sin\frac{(n+\frac{1}{2})\pi x}{l}\cdot x\ dx = (-1)^n\frac{2l}{(n+\frac{1}{2})^2\pi^2}$.

5(a). Let $u(x,t) = X(x)T(t)$, then

$$-X''(x) = \lambda X(x),$$

$$X(0) = 0,\ X'(l) = 0.$$

By Theorem 3, there is no negative eigenvalue. It is easy to check that 0 is not an eigenvalue. Hence, there are only positive eigenvalues.
Let $\lambda = \beta^2$, $\beta > 0$, then we have

$$X(x) = A\cos\beta x + B\sin\beta x.$$

Hence the bounndary condtions imply

$$A = 0,\ B\beta\cos\beta l = 0.$$

$$\beta = \frac{(n+\frac{1}{2})\pi}{l},\ n = 0, 1, 2, \ldots$$

So the eigenfunctions are

$$X_n(x) = \sin\frac{(n+\frac{1}{2})\pi x}{l},\ n = 0, 1, 2, \ldots \qquad \square$$

6. Let $X'(x) = \lambda X(x)$, $\lambda \in \mathbb{C}$, then

$$X(x) = e^{\lambda x}.$$

By the boundary condition $X(0) = X(1)$, we have

$$e^{\lambda} = 1.$$

Hence,

$$\lambda_n = 2n\pi i,\ X_n(x) = e^{2n\pi x i},\ n \in \mathbb{Z}.$$

Since, if $m \neq n$,

$$\int_0^1 X_n(x)\overline{X_m(x)}dx = \int_0^1 e^{2(n-m)\pi x i}dx = 0.$$

Therefore, the eigenfunctions are orthogonal on the interval $(0, 1)$. $\square$

8. If

$$X_1'(a) - a_a X_1(a) = X_2'(a) - a_a X_2(a) = 0,$$

and

$$X_1'(b) + a_b X_1(b) = X_2'(b) + a_b X_2(b) = 0,$$

then

$$\begin{aligned}(-X_1'X_2 + X_1X_2')|_a^b &= -X_1'(b)X_2(b) + X_1(b)X_2'(b) + X_1'(a)X_2(a) - X_1(a)X_2'(a)\\ &= a_bX_1(b)X_2(b) - X_1(b)a_bX_2(b) + a_aX_1(a)X_2(a) - X_1(a)a_aX_2(a) = 0. \qquad \square\end{aligned}$$

9. For $j = 1, 2$, suppose that

$$\begin{aligned}X_j(b) &= \alpha X_j(a) + \beta X_j'(a)\\ X_j'(b) &= \gamma X_j(a) + \delta X_j'(a).\end{aligned}$$

Then,

$$\begin{aligned}(X_1'X_2 - X_1X_2')|_a^b &= X_1'(b)X_2(b) - X_1(b)X_2'(b) - X_1'(a)X_2(a) + X_1(a)X_2'(a)\\ &= [\gamma X_1(a) + \delta X_1'(a)][\alpha X_2(a) + \beta X_2'(a)]\\ &\quad - [\alpha X_1(a) + \beta X_1'(a)][\gamma X_2(a) + \delta X_2'(a)] - X_1'(a)X_2(a) + X_1(a)X_2'(a)\\ &= (\alpha\delta - \beta\gamma - 1)X_1'(a)X_2(a) + (1 + \beta\gamma - \alpha\delta)X_1(a)X_2'(a)\\ &= (\alpha\delta - \beta\gamma - 1)(X_1X_2)'|_{x=a}.\end{aligned}$$

Therefore, the boundary conditions are symetric if and only if $\alpha\delta - \beta\gamma = 1$. $\square$

10. (a)(By induction)First, it is easy to check that Z_2 is orthogonal to Z_1. Assume that $Z_1, Z_2, ..., Z_n$ are orthogonal to each other, and, by definition, $Y_{n+1} = X_{n+1} - \sum_{k=1}^{n}(X_{n+1}, Z_k)Z_k$. Thus, by assumption, for $l = 1, 2, ..., n$,

$$(Z_l, Z_{n+1}) = ((Z_l, X_{n+1}) - \sum_{k=1}^{n}(X_{n+1}, Z_k)(Z_l, Z_k))/||Y_{n+1}|| = ((Z_l, X_{n+1}) - (Z_l, X_{n+1}))/||Y_{n+1}|| = 0,$$

$(Z_{n+1}, Z_{n+1}) = 1$. That is $Z_1, Z_2, ..., Z_{n+1}$ are orthogonal to each other.

(b)

$$Z_1 = \frac{cosx + cos2x}{\sqrt{\int_0^\pi (cosx + cos2x)^2 dx}} = (cosx + cos2x)/\sqrt{\pi},$$

$$Y_2 = 3cosx - 4cos2x - Z_1 \int_0^\pi (3cosx - 4cos2x)Z_1 dx = 7(cosx - cos2x)/2,$$

$$Z_2 = \frac{Y_2}{\sqrt{\int_0^\pi Y_2^2 dx}} = (cosx - cos2x)/\sqrt{\pi}.$$

12. By the divergence theorem,

$$f'g|_a^b = \int_a^b (f'(x)g(x))'dx = \int_a^b f''(x)g(x) + f'(x)g'(x)dx,$$

$$\int_a^b f''(x)g(x)dx = -\int_a^b f'(x)g'(x)dx + f'g|_a^b. \qquad \square$$

13. Substitute $f(x) = X(x) = g(x)$ in the Green's first identity, we have

$$\int_a^b X''(x)X(x)dx = -\int_a^b X'^2(x)dx + (X'X)|_a^b \le 0.$$

Since $-X'' = \lambda X$, so

$$-\lambda \int_a^b X^2(x)dx \le 0.$$

Therefore, we get $\lambda \ge 0$ since $X \not\equiv 0$. $\square$

Exercise 5.4

1. The partial sum is given by

$$S_n = \frac{1 - (-1)^n x^{2n}}{1 + x^2}.$$

   (a) Obviously for any x_0 fixed, $S_n \to \frac{1}{1+x_0^2}$. Thus it converges to $\frac{1}{1+x^2}$ pointwise.

   (b) Let $x_n = 1 - \frac{1}{n}$, then $x^{2n} \to e^{-2}$. Thus it does not converge uniformly.

   (c) It will converge to $S(x) = \frac{1}{1+x^2}$ in the L^2 sence since

$$\begin{aligned}\int_{-1}^{1} |S_n - S|^2 dx &= \int_{-1}^{1} \frac{x^{4n}}{(1+x^2)^2}dx \\ &\le \int_{-1}^{1} x^{4n} dx \\ &\le \frac{2}{4n+1} \to 0 \quad \text{as } n \to \infty. \qquad \square\end{aligned}$$

2. This is an easy consequence combined Theorem 2 and Theorem 3 on Page 124 and Theorem 4 on Page 125. $\square$

3. (a) For any fixed point x_0, WLOG, we assume $x_0 < \frac{1}{2}$. Then there is N_0 such that for $n > N_0$,

$$x_0 < \frac{1}{2} - \frac{1}{n},$$

which implies that $f_n(x_0) \equiv 0$. Thus $f_n(x) \to 0$ pointwisely.

(b) Let $x_n = \frac{1}{2} - \frac{1}{n}$, then $f_n(x_n) = -\gamma_n \to -\infty$, which implies that the convergence is not uniform.

(c) By direct computation, we have

$$\int f_n^2(x)dx = \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} \gamma_n^2 dx + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} \gamma_n^2 dx = \frac{2\gamma_n^2}{n}.$$

For $\gamma_n = n^{\frac{1}{3}}$,

$$\int f_n^2(x)dx = 2n^{-\frac{1}{3}} \to 0 \quad \text{as } n \to \infty.$$

(d) By the computation in (c), for $\gamma_n = n$,

$$\int f_n^2(x)dx = 2n \to \infty \quad \text{as } n \to \infty. \qquad \square$$

4. For odd n,

$$\int_{\frac{1}{4}-\frac{1}{n^2}}^{\frac{1}{4}+\frac{1}{n^2}} 1^2 dx = \frac{2}{n^2} \to 0.$$

For even n,

$$\int_{\frac{3}{4}-\frac{1}{n^2}}^{\frac{3}{4}+\frac{1}{n^2}} 1^2 dx = \frac{2}{n^2} \to 0.$$

Thus, for any n,

$$\|g_n(x)\|_{L^2}^2 = \frac{2}{n^2} \to 0 \quad \text{as} \quad n \to \infty. \qquad \square$$

5. (a) We see that $A_0 = \frac{2}{3}\int_1^2 dx = \frac{4}{3}$ and $A_m = \frac{2}{3}\int_2^3 \cos\frac{m\pi x}{3} dx = -\frac{2}{mx}\sin\frac{m\pi}{3}$. So, the first four nonzero terms are $\frac{4}{3}$, $-\frac{\sqrt{3}}{pi}\cos\frac{\pi x}{3}$, $-\frac{\sqrt{3}}{2\pi}\cos\frac{2\pi x}{3}$ and $\frac{\sqrt{3}}{4\pi}\cos\frac{4\pi x}{3}$.

(b) We can express $\phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos\frac{n\pi x}{3} + B_n \sin\frac{n\pi x}{3})$. by Theorem 4 of Sectiion 4, since $\phi(x)$ and its derivative is piecewise continuous, so we get the fourier series will converge to $f(x)$ except at $x = 1$, while the value of this series at $x = 1$ is $\frac{1}{2}$.

(c) By corollary 7, we see that it converge to $\phi(x)$ in L^2 sense.

(d) Put $x = 0$, we see that the sine series vanish, it turns out to be that $\phi(0) = \frac{2}{3} - \frac{\sqrt{3}}{\pi}\sum_{1\le m<\infty, m\neq 3n} \frac{(-1)^{[\frac{m}{3}]}}{m}\cos\frac{m\pi 0}{3}$ thus we obtain the sum of thee series is $\frac{2\pi}{3\sqrt{3}}$. $\square$

6. The series is $\cos x = \sum_{n=1}^{\infty} a_n \sin nx$. If $n > 1$,

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx dx = -\frac{1}{\pi}[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}]\Big|_0^{\pi} = \frac{2n(1+(-1)^n)}{(n^2-1)\pi}.$$

If $n = 1$, $a_1 = 0$. The sum series is 0 if $x = -\pi, 0, \pi$. By Theorem 4 in Section 4, the sum series converges to $\cos x$ pointwisely in $0 < x < \pi$, and to $-\cos x$ for $-\pi < x < 0$. $\square$

7. (a) Obviously $\phi(x)$ is odd. Thus, its full Fourier series is just the Sine Fourier series, i.e.

$$\sum_{n=1}^{\infty} B_n \sin n\pi x,$$

where B_n satisfies

$$B_n = \int_{-1}^{1} \phi(x) \sin n\pi x dx = \frac{2}{n\pi}.$$

(b) By (a), the first three nonzero terms are

$$\frac{2}{\pi} \sin \pi x,\ \frac{1}{\pi} \sin 2\pi x,\ \frac{2}{3\pi} \sin 3\pi x.$$

(c) Since

$$\int_{-1}^{1} |\phi(x)|^2 dx = 2 \int_{0}^{1} (1-x)^2 dx \leq 2,$$

it cconverges in the mean square sense according to Corollary 7.

(d) Since $\phi(x)$ is continuous on $(-1, 1)$ except at the point $x = 0$. Therefore, Theorem 4 in Section 4 implies it converges pointwisely on $(-1, 1)$ expect at $x = 0$.

(e) Since the series does not converge pointwise, it does not converge uniformly.

8. (a)$f(x) = x^3, f'(x) = 3x^2, f''(x) = 6x$ exist and continuous on $[0, l]$, $f(0) = 0, f(l) = l^3 \neq 0$ and $\int_0^l x^6 dx = l^7/7$ is finite, thus, the Fourier sine series of $f(x)$ converges pointwise on $(0, l)$ and in the mean square sense but not uniformly.

(b)$f(x) = lx - x^2, f'(x) = l - 2x, f''(x) = -2$ exist and continuous on $[0, l]$, $f(0) = f(l) = 0$ and $\int_0^l (lx - x^2)^2 dx = l^5/30$ is finite, thus, the Fourier sine series of $f(x)$ converges pointwise, uniformly on $[0, l]$ and in the mean square sense.

(c)$f(x) = x^{-2}, f'(x) = -2x^{-3}, f''(x) = 6x^{-4}$ exist and continuous on $(0, l)$, do not exist when $x = 0$ and $\int_0^l x^{-4} dx$ is not finite, thus, the Fourier sine series of $f(x)$ converges pointwise on $(0, l)$ but not in the mean square sense nor uniformly.

Exercise 5.6

1. (a) (Use the method of shifting the data.)
Let $v(x, t) := u(x, t) - 1$, then v solves

$$v_t = v_{xx},\ v_x(0, t) = v(1, t) = 0,\ \text{and}\, v(x, 0) = x^2 - 1.$$

By the method of seperation of variables, we have

$$v(x, t) = \sum_{n=0}^{\infty} A_n e^{-(n+\frac{1}{2})^2\pi^2 t} \cos[(n + \frac{1}{2})\pi x],$$

where

$$A_n = (-1)^{n+1} 4(n + \frac{1}{2})^{-3} \pi^{-3}.$$

Hence,

$$u(x, t) = 1 + \sum_{n=0}^{\infty} A_n e^{-(n+\frac{1}{2})^2\pi^2 t} \cos[(n + \frac{1}{2})\pi x],$$

where A_n is as before.

(b) 1. $\square$

2. In the case $j(t) = 0$ and $h(t) = e^t$, by (10) and the initial condition $u_n(0) = 0$,
$$u_n(t) = \frac{2n\pi k}{(\lambda_n k + 1)l^2}(e^t - e^{-\lambda_n kt}).$$
Therefore,
$$u(x,t) = \sum_{n=1}^{\infty} \frac{2n\pi k}{(\lambda_n k+1)l^2}(e^t - e^{-\lambda_n kt}) \sin\frac{n\pi x}{l}. \qquad \square$$

5. It is easy to check that $\dfrac{e^t \sin 5x}{1+25c^2}$ solves
$$v_t t = c^2 v_{xx} + e^t \sin 5x, \qquad \text{and} \qquad v(0,t) = v(\pi,t) = 0.$$
Using the method of shifting the data, we have
$$u(x,t) = \frac{e^t \sin 5x}{1+25c^2} + \sum_{n=1}^{\infty}(A_n \cos(nct) + B_n \sin(nct))\sin(nx),$$
where
$$A_n = -\frac{2}{\pi}\int_0^{\pi} \frac{1}{1+25c^2}\sin 5x \sin nx\; dx = \begin{cases} -\dfrac{1}{1+25c^2} & n=5 \\ 5 & \text{otherwise} \end{cases};$$
$$\begin{aligned} B_n &= \frac{2}{nc\pi}\int_0^{\pi}[\sin 3x - \frac{1}{1+25c^2}\sin 5x]\sin nx\; dx \\ &= \begin{cases} \frac{1}{3c} & n=3 \\ -\dfrac{1}{5c(1+25c^2)} & n=5 \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$
So the formula of the solution can be simplfied as
$$u(x,t) = \frac{1}{3c}\sin 3ct \sin 3x + \frac{1}{1+25c^2}\left(e^t - \cos 5ct - \frac{1}{5c}\sin 5ct\right)\sin 5x. \qquad \square$$

8. (Expansion Method) Let
$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} u_n(t)\sin\frac{n\pi x}{l}, \\ \frac{\partial u}{\partial t}(x,t) &= \sum_{n=1}^{\infty} v_n(t)\sin\frac{n\pi x}{l}, \\ \frac{\partial^2 u}{\partial x^2}(x,t) &= \sum_{n=1}^{\infty} w_n(t)\sin\frac{n\pi x}{l}. \end{aligned}$$
Then
$$\begin{aligned} v_n(t) &= \frac{2}{l}\int_0^l \frac{\partial u}{\partial t}\sin\frac{n\pi x}{l}dx = \frac{du_n}{dt}, \\ w_n(t) &= \frac{2}{l}\int_0^l \frac{\partial^2 u}{\partial x^2}\sin\frac{n\pi x}{l}dx = \frac{du_n}{dt}, \\ &= -\frac{2}{l}\int_0^l (\frac{n\pi}{l})^2 u(x,t)\sin\frac{n\pi x}{l}dx + \frac{2}{l}(u_x \sin\frac{n\pi x}{l} - \frac{n\pi}{l}u\cos\frac{n\pi x}{l})\Big|_0^l \\ &= -\lambda_n u_n(t) - 2n\pi l^{-2}(-1)^n At, \end{aligned}$$

where $\lambda_n = (n\pi/l)^2$. Here we used the Green's second identity and the boundary conditions. Hence, by the PDE $u_t = ku_{xx}$ and the initial condition $u(x,0) = 0$, we get

$$\frac{du_n}{dt} = k[-\lambda_n u_n(t) - 2n\pi l^{-2}(-1)^n At],$$

$$u_n(0) = 0.$$

Hence,

$$u_n(t) = (-1)^{n+1} 2n\pi l^{-2} A[\frac{t}{\lambda_n} - \frac{1}{\lambda_n^2 k} + \frac{e^{-\lambda_n kt}}{\lambda_n^2 k}].$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} (-1)^{n+1} 2n\pi l^{-2} A[\frac{t}{\lambda_n} - \frac{1}{\lambda_n^2 k} + \frac{e^{-\lambda_n kt}}{\lambda_n^2 k}] \sin \frac{n\pi x}{l},$$

where $\lambda_n = (n\pi/l)^2$.

11. The general solution is $y_n(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$, where $y_1(t) = e^{c\sqrt{-\lambda_n}t}, y_2(t) = e^{-c\sqrt{-\lambda_n}t}$ are a fundamental set of solutions and $Y(t) = -y_1(t) \int_0^t \frac{y_2(s)g(s)}{W(y_1,y_2)(s)} ds + y_2(t) \int_0^t \frac{y_1(s)g(s)}{W(y_1,y_2)(s)} ds$ (here $W(y_1,y_2)(s) = y_1(s)y_2'(s) - y_1'(s)y_2(s) = -2c\sqrt{\lambda_n} \neq 0$ if $\lambda_n \neq 0$, $g(s) = -2n\pi l^{-2}((-1)^n k(s) - h(s)) + f_n(s))$. The constant c_1 and c_2 are determined by the initial conditions.