Tutorial 2

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1. Boundary conditions for sound.

Small disturbances are governed by

$$\begin{aligned} \frac{\partial \mathbf{v}}{dt} + \frac{c_0^2}{\rho_0}\nabla\rho = 0 \\ \frac{\partial \rho}{dt} + \rho_0 \mathrm{div}\mathbf{v} = 0 \end{aligned} \tag{1}$$

where $\mathbf{v}$ and ρ as unknowns are velocity and density. ρ_0 is the density and c_0 is the speed of sound in still air.

Assume that the is no swirl of the sound, that is, curl$\mathbf{v} = 0$. Then (1) can be written as

$$\begin{aligned} \frac{\partial^2 \mathbf{v}}{dt^2} = c_0^2 \Delta \mathbf{v} \\ \frac{\partial^2 \rho}{dt^2} = c_0^2 \Delta \rho_0 \end{aligned} \tag{2}$$

Boundary conditions:

Case 1: $\mathbf{v} \cdot \mathbf{n} = 0$: the sound propagates in a closed sound-insulated room with rigid wall. Furthermore, curl$\mathbf{v} = 0$ implies that there exists a scaler function ϕ such that $v = \nabla\phi$. Thus this boundary condition turns to $\frac{\partial \phi}{\partial \mathbf{n}} = 0$, Neumann boundary conditon for ϕ.

Case 2: $\rho = \rho_0$: at an open window. Dirichlet Boundary condition for ρ.

Case 3: $\mathbf{v} \cdot \mathbf{n} = a(\rho - \rho_0)$: at a soft wall.

A different kind of boundary condition in the case of the wave equation is

$$\frac{\partial u}{\partial n} + b\frac{\partial u}{\partial t} = 0.$$

This condition means that energy is radiated to ($b > 0$) or absorbed from ($b < 0$) the exterior through the boundary. (To be proved later by Energy method)

2. Condition at infinity.

When the domain is unbounded, what kind of the condition should we impose? The physics usually provides conditions at infinity.

For example, consider the Schrodinger equation where the domain is the whole space. Since

$$\iiint_{\mathbb{R}^n} |u|^2 dxdydz = 1$$

we know that u "vanishes at infinity", that is,

$$\lim_{r\to+\infty} u(x,y,z,t) = 0$$

where r is the spherical coordinates.

3. Classify the following equations:

a. $u_{xx} - 5u_{xy} = 0$

b. $4u_{xx} - 12u_{xy} + 9u_{yy} + u_y = 0$

c. $4u_{xx} + 6u_{xy} + 9u_{yy} = 0$

Solution: a. $a_{12}^2 - a_{11}a_{22} = (-\frac{5}{2})^2 > 0$, hyperbolic;
b. $a_{12}^2 - a_{11}a_{22} = (-6)^2 - 4\cdot 9 = 0$, parabolic;
a. $a_{12}^2 - a_{11}a_{22} = 3^2 - 4\cdot 9 < 0$, elliptic;

4. Check that

$$\sum_{i,j} b_{ki}a_{ij}b_{lj} = (BAB^t)_{k,l}$$

where a_{ij} means the ij-element of the matrix A.

Since $(AB)_{i,j} = \sum_l a_{il}b_{lj}$, then $(BAB^t)_{k,l} = \sum_j (BA)_{k,j}B^t_{j,l} = \sum_j (BA)_{k,j}b_{lj} = \sum_j (\sum_i b_{ki}a_{ij})b_{lj} = \sum_{i,j} b_{ki}a_{ij}b_{lj}$.