Suggested Solution to Quiz 1

Feb 14, 2017

1. (5 points) For each of the following equations, state the order, type and whether it is nonlinear, linear inhomogeneous, or linear homogeneous:

   (a) $4\partial_t u - \partial_x^2 u + 1 = 0$
   (b) $\partial_t^2 u - \partial_x^2 u + u^2 = 0$
   (c) $\partial_{xy}^2 u = \sin^2(4x) + 1$
   (d) $2\partial_x^2 u + \partial_{xy}^2 u + \partial_y^2 u = 0$

   Solution:

   (a) 2nd order(0.25points); parabolic equation(0.5points); linear inhomogeneous(0.5points).
   (b) 2nd order(0.25points); hyperbolic equation(0.5points); nonlinear(0.5points)
   (c) 2nd order(0.25points); hyperbolic equation(0.5points); linear inhomogeneous(0.5points).
   (d) 2nd order(0.25points); elliptic equation(0.5points); linear homogeneous(0.5points).

2. (5 points) Solve the equation $\partial_x u + x\partial_y u = 0$ with the following two conditions:

   (a) $u(0, y) = y^2$.
   (b) $u(x, 0) = x^2$

   Solution: The characteristic curves satisfy the ODE:

   $$\frac{dx}{1} = \frac{dy}{x}$$

   Hence the characteristic curves are

   $$y = \frac{1}{2}x^2 + C \quad \text{(1point)}$$

   Therefore, the general solution is

   $$u(x, y) = f(y - \frac{1}{2}x^2) \quad \text{(1point)}$$

   where f is an arbitrary function.

   (a) By $u(0, y) = y^2$, we have $u(0, y) = f(y) = y^2$.Therefore $u(x, y) = (y - \frac{1}{2}x^2)^2$ on $\mathbb{R}^2$.(1 point)
   (b) By $u(x, 0) = x^2$, we have $u(x, 0) = f(-\frac{1}{2}x^2) = x^2$ which implies $f(z) = -2z$. Therefore $u(x, y) = -2(y - \frac{1}{2}x^2) = x^2 - 2y$ (1 point) on the domain $\{(x, y) : y - \frac{1}{2}x^2 \le 0\}$(1 point).On the domain $\{(x, y) : y - \frac{1}{2}x^2 > 0\}$, the solution of $u(x, y)$ can not be determined uniquely.

3. (5points) Is the backward heat equation well-posed?

   $$\begin{cases} \partial_t u = \partial_x^2 u, \quad -\infty < x < \infty, \quad t < 0 \\ u(x, t = 0) = \phi(x) \end{cases}$$

Why?

Solution: No. (1point)

Suppose that u is a solution to

$$\begin{cases} \partial_t u = \partial_x^2 u, & -\infty < x < \infty, \quad t < 0 \\ u(x, t = 0) = \phi(x) \end{cases}$$

then $u_n(x,t) = u + \frac{1}{n} \sin nx e^{-n^2 t}$ solves the following probem

$$\begin{cases} \partial_t u = \partial_x^2 u, & -\infty < x < \infty, \quad t < 0 \\ u(x, t = 0) = \phi(x) + \dfrac{1}{n} \sin nx \end{cases}$$

for all positive integer n.

On one hand, $\max_{-\infty<x<\infty} |\frac{1}{n} \sin(nx)| \to 0$ as $n \to \infty$, that is, the initial data change a little.
On the other hand, when $t = -1$, $|\frac{1}{n} \sin(nx) e^{n^2}| \to +\infty$ as $n \to \infty$ except for a few x.
This violates the stability in the uniform sense. (4points)