§10.4.2 Plane waves and formation of shock waves

Given any direction $w \in \mathbb{R}^m$, $|w| = 1$, look for special

$$u(x,t) = U(x \cdot w, t)$$
$$\xi = x \cdot w, \qquad u(x,t) = U(\xi, t)$$
$$\partial_t u + \sum A_j \, \partial_{x_j} u = 0$$

$\Rightarrow$

$$\partial_t U + A(u,w)\, \partial_\xi U = 0$$

where

$$A(u,w) = \sum_{j=1}^{m} A_j(u) w_j$$

$$\begin{cases} \partial_t u + A(u,w)\, \partial_\xi u = 0, \quad t > 0, \quad \xi \in \mathbb{R}^1 \\ u(x, t=0) = u_0(\xi) \end{cases}$$

P. Lax, F. John, L. Hörmander.

§10.4.3 Shock wave formation in plane wave solution

$$\begin{cases} \partial_t u + \displaystyle\sum_{j=1}^{m} \partial_{x_j} F_j(u) = 0 \\ u(x, t=0) = u_0(x) \end{cases} \tag{10.38}$$

$u(x,t) = U(x \cdot w, t)$ for a given direction $w \in \mathbb{R}^m$, $|w| = 1$, $A_i(u) = \frac{\partial F_i}{\partial u} = \cap v u$.
Set $\xi = x \cdot w$,

$$\begin{cases} \partial_t u + A(u,w)\, \partial_\xi u = 0 \\ u(\xi, t=0) = u_0(\xi) = u_0(x \cdot w) \end{cases} \tag{10.39}$$

where $A(u,w) = \displaystyle\sum_{i=1}^{m} w_i A_i(u)$,

$$\begin{array}{ccccccc} \lambda_1(u,w) & \le & \lambda_2(u,w) & \le & \cdots & \le & \lambda_n(u,w) \\ r_1(u,w), & & r_2(u,w), & & \cdots & & r_n(u,w) \\ l_1(u,w), & & l_2(u,w), & & \cdots & & l_n(u,w) \end{array}$$

$LR = I$.

<u>Blow-up of simple waves</u>

Let k be fixed, $1 \le k \le n$. Assume that $\bar{u}_0 \in D \subset \mathbb{R}^n$. Regard $r_k(u)$ as a vector field on D. As we can look the integral curve of $r_k(u)$ through $\bar{u}_0$, i.e.

$$\begin{cases} \dfrac{dU_k(\sigma)}{d\sigma} = r_k(U_k(\sigma)) \\ U_k(\sigma = 0) = \bar{u}_0 \end{cases} \tag{10.40}$$

$\exists\, \sigma_\pm$, $\sigma_- < \sigma < \sigma_+$ such that (10.40) has a smooth solution $U_k(\sigma)$, $\sigma \in (\sigma_-, \sigma_+)$.
$U_k(\sigma)$ is called a k-th wave curve through $\bar{u}_0$.

Next, solve the following initial value problem

$$\begin{cases} \partial_t \sigma + \lambda_k(U_k(\sigma)) \partial_\xi \sigma = 0 & \xi \in \mathbb{R}^1, \quad t > 0 \\ \sigma(t=0) = \sigma_0(\xi) & \sigma_- < \sigma_0(\xi) < \sigma_+, \quad \forall \xi \in \mathbb{R}^1 \end{cases} \tag{10.41}$$

$\sigma(\xi,t)$ exist locally on $[0,T]$, T is maximal time.
Set

$$U(\xi,t) = U_k(\sigma(\xi,t)) \tag{10.42}$$

Claim: $U(\xi,t)$ defined by (10.42), is a solution to the equation in (10.39).

$$\begin{aligned} \partial_t u &= \frac{Du_k}{D\sigma}\partial_t\sigma = r_k(u_k)\partial_t\sigma \\ \partial_\xi u &= \partial_\xi\sigma\, r_k(u_k) \\ \partial_t u + A(u)\partial_\xi u &= \partial_t\sigma\cdot r_k(u_k) + A(u_k)r_k(u_k)\partial_\xi\sigma \\ &= (\partial_t\sigma + \lambda_k(u_k)\partial_\xi\sigma)r_k(u_k) = 0 \end{aligned}$$

Definition 10.7 *The $u_k(\sigma(\xi,t))$ defined by (10.42) is called a simple wave. Recall the previous result on the formation of shocks that (10.41) has a global smooth solution iff*

$$\frac{d}{d\xi}\lambda_k(u_k(\sigma_0(\xi))) \geq 0$$

In other words, if $\exists\,\xi_0 \in \mathbb{R}^1$, such that

$$\frac{d}{d\xi}\lambda_k(u_k(\sigma_0(\xi)))|_{\xi=\xi_0} < 0 \tag{10.43}$$

shock must form in finite time

$$\begin{aligned} \frac{d}{d\xi}\lambda_k(u_k(\sigma_0(\xi))) &= \nabla\lambda_k\cdot\frac{du_k}{d\sigma}\frac{d\sigma_0}{d\xi} \\ &= (\nabla\lambda_k\cdot r_k)\frac{d\sigma_0}{d\xi} \end{aligned}$$

Definition 10.8 *(P. D. Lax) The k-th characteristic field is said to be genuinely nonlinear at $u_0 \in \mathcal{D}$ in the direction w, if*

$$(\nabla\lambda_k\cdot r_k)(u_0) \neq 0 \tag{10.44}$$

And the k-th field is said to be linearly degenerate if

$$(\nabla\lambda_k\cdot r_k)(u) \equiv 0 \qquad \forall u \in B_\delta(u_0)$$

Proposition 10.6 *Assuming that the system in (10.38) is not linearly degenerate in the direction w. Then $\exists$ a k-simple wave which blow-up in finite time, which is determined by*

$$\frac{\sigma_0'(\xi)}{1 + (\partial_\xi\lambda_k(u_k(\sigma_0(\xi))))t}$$

Next, blow-up results due to F. John.

$$\begin{cases} \partial_t u + A(u)\,\partial_\xi u = 0 \\ u(\xi, t=0) = u_0(\xi) \end{cases}$$

u_0 has compact support.

Theorem 10.9 *(F. John) Assume that*

(i) The system in (10.39) is genuinely nonlinear on $B_\delta(\bar{u}_0)$.

(ii) $u_0 \in H^s_{ul}(\mathbb{R}^1)$, $s > 3$. u_0 *has compact support in the sense that*

$$u_0 - \bar{u}_0 \in C_0^2(\mathbb{R}^1) \quad \mathrm{supp}\,(u_0 - \bar{u}_0) \subset [a, b]$$

Then there exists a $\theta_0 = \theta_0(\delta, A) > 0$ *such that if*

$$0 < \theta = (b-a)^2\,|u_0^{''}|_{L^\infty} \le \theta_0$$

Then the solution to (10.39) must form shocks in finite time.

Key ideas of the proof:

• Huygen's principle

If $A(u) = A_0$, constant matrix

$$\lambda_1, \cdots, \lambda_n \qquad \text{constant}$$

$u(x,t) = \displaystyle\sum_{i=1}^{m} \alpha_i\, r_i,$

- characteristic decomposition of spatial derivatives
- reduced to a Ricatti equation

Step 1: Canonical representation

Let $u(\xi, t)$ be a C^2-smooth solution. Consider the j-th characteristic $\xi = \xi_j(t)$, i.e.

$$\frac{d\xi_j}{dt} = \lambda_j(u(\xi_j(t), t))$$

We denote the differentiation along the j-th characteristic as $\dfrac{d}{dt_j}$, i.e.

$$\frac{d}{dt_j} = \partial_t + \lambda_j\,\partial_\xi$$

Then the system (10.39) can be written as

$$l_j^t(u)\frac{d}{dt_j}u = 0 \qquad j = 1, \cdots, n \tag{10.45}$$

(10.45) is called a canonical representation of (10.39).

Step 2: Characteristic decomposition $\partial_\xi u$

$$\partial_\xi u = \sum_{i=1}^{n} w_i\, r_i(u) \tag{10.46}$$

where $w_i = l_i^t(u)\,\partial_\xi u$.

John's formula

$$\frac{D}{Dt_i} w_i = \sum_{k,l=1}^{n} \gamma_{ikl}\, w_k\, w_l \tag{10.47}$$

$\gamma_{ikl}(u)$ are called interaction coefficients given by

$$\gamma_{ikl} = -\frac{1}{2}(\lambda_k - \lambda_l)\, l_i[r_k, r_l] - (\nabla\lambda_i \cdot r_k)\delta_{il} \tag{10.48}$$

$$[r_k, r_l] = \nabla r_k \cdot r_l - \nabla r_l \cdot r_k$$

Properties of γ_{ikl}

$$\begin{cases} (1) & \gamma_{iii} = -\nabla\lambda_i \cdot r_i = -1 \quad \text{(by normalization)} \\ (2) & \gamma_{ikk} = 0 \quad \text{if } i \neq k \end{cases} \tag{10.49}$$

Key idea:

(1) "major" term in (10.47) is $\gamma_{iii}\, w_i^2 = -w_i^2$.

(2) (10.49) implies that no other self-interactions in (10.46), i.e. all the other terms in (10.46) involves $w_j\, w_k$, $j \neq k$ which are the products of waves from different family.

(3) For the initial data with compact support, the approximate Huygen's principle applies, so waves with different speeds eventually separate, thus $w_k\, w_l$ must become smaller for large time, so

$$\frac{d}{dt_i} w_i = \gamma_{iii}\, w_i^2 + O(1)$$

Thus, one can obtain a Ricatti type differential inequality, D_0 blow-up in finite time for w_i.
In order to ensure the u still remains $B_\delta(0)$, then one has to show $||\partial_\xi u||_{L^1}$ is bounded.

Remark 10.19 *In Theorem 10.9, we require that every characteristic family is genuinely nonlinear, which does not apply to 3×3 gas dynamics equation since for which the entropy wave family is always linearly degenerate.*

Theorem 10.10 *(JDE, 1979, T. P. Liu) Assume that*

(i) The system in (10.39) is strictly hyperbolic.

(ii) Each characteristic field is either genuinely nonlinear or linearly degenerate, $\exists\, N \subset \{1, 2, \cdots, n\}$, such that λ_i is genuinely nonlinear if $i \in N$, λ_j is linearly degenerate if $j \in N^c = \{1, 2, \cdots, n\} \setminus N$.

(iii) Linear waves never generate nonlinear waves, i.e.

$$\gamma_{ikl} = 0 \qquad if \qquad i \in N \qquad and \qquad k, l \in N^c \tag{10.50}$$

(iv) $u_0 \in H^s_{ul}(\mathbb{R}^1)$, $s > 3$, $u - \bar{u}_0 \in C^1(\mathbb{R}^1)$, $\operatorname{supp}(u - \bar{u}_0) \subset [a, b]$.

Then there exists $\theta_0 = \theta_0(\delta, A) > 0$, such that if

$$\theta = (b - a)\, |u_0'|_{L^\infty} \leq \theta_0$$

$$0 < \varepsilon = \max_{i \in N} |w_i(\xi)|_{L^\infty}, \quad w_i(\xi) = l_i^t(u_0(\xi))\, \partial_\xi\, u_0(\xi) \tag{10.51}$$

Then any C^1-solution to problem (10.39) forms shocks in finite time. Furthermore, if $\theta \leq \theta_0$, $\varepsilon = 0$, then smooth solution exists globally.

Remark 10.20 *If N^c contains only one element, then (10.50) is satisfied automatically. However, for one-dimensional gas dynamics, only one family (entropy wave family) is linearly degenerate. So Theorem 10.10 indeed applies to 3×3 gas dynamics system.*

Remark 10.21 *In (10.51), ε measure the strength of the initial nonlinear waves, Theorem 10.10 implies if no nonlinear waves initially, the global smooth solution exists. In particular, if the system is totally linearly degenerate, i.e. $N = \phi$. Then (10.50) is satisfied automatically also. Theorem 10.10 implies global existence of smooth solutions. How about the multi-d case?*

Remark 10.22 *All the results of F. John has been generalized to the case, the characteristic fields may have inflection points, by Hirmander, Da-Tsien Li, etc.*

<u>Shock formation for system endowed with a coordinates of Riemann invariants</u>

Definition 10.9 *A $c(u)$ is said to be an i-Riemann invariant if*

$$\nabla c(u) \cdot r_i(u) \equiv 0 \qquad \forall u \in \mathcal{D} \tag{10.52}$$

Look at (10.52), which is a 1-st order PDE. By the characteristic method, one can find $(n-1)$ i-th Riemann invariants $c_j(u)$, $j = 1, \cdots, n$, $j \neq i$, such that

$$\nabla c_j \cdot r_i = 0$$

and ∇c_j, $j \neq i$, span the orthogonal complement of r_i.

Definition 10.10 *The system*

$$\partial_t u + A(u)\, \partial_\xi u = 0 \tag{10.53}$$

is said to be endowed with a coordinate system of Riemann invariants, if $\exists\, n$ scalar valued function $c_1(u), \cdots, c_n(u)$ such that $c_j(u)$ is an i-th Riemann invariants for (10.53) for all $j \neq i$, $i, j = 1, \cdots, n$, and $\nabla c_i(u)$, $i = 1, \cdots, n$ are linearly independent.

Proposition 10.7 *The functions $(c_1(u), \cdots, c_n(u))$ form a coordinate system of Riemann invariants of (10.53) iff*

$$\nabla c_i(u) \cdot r_j(u) = \begin{cases} 0 & i \neq j \\ \neq 0 & i = j \end{cases} \tag{10.54}$$

Since (10.54) $\Rightarrow c_i(u) // l_i(u)$, therefore

$$(\nabla c_1(u), \cdots, \nabla c_n(u))^T = L(u)$$

Remark 10.23 *$\nabla c_i(u)$ must be a left eigenvector of $A(u)$ associated with λ_i.*

Recall the canonical form of (10.53)

$$l_i(u)(\partial_t u + \lambda_i\, \partial_\xi u) = 0, \qquad i = 1, \cdots, n \tag{10.55}$$

Now assume that (10.53) is endowed with a coordinate system of Riemann invariants

$$c(u) = (c_1(u), \cdots, c_n(u))$$

Then

$$l_i(u) = \nabla c_i(u)$$

Then go back to (10.55)

$$\begin{aligned} 0 &= l_i(u)(\partial_t u + \lambda_i(u)\,\partial_\xi u) = \nabla c_i(u)(\partial_t u + \lambda_i(u)\,\partial_\xi u) \\ &= \partial_t\, c_i(u) + \lambda_i(u)\,\partial_\xi\, c_i(u) \end{aligned}$$

$$\partial_t\, c_i + \lambda_i(c)\,\partial_\xi\, c_i = 0 \qquad i = 1, 2, \cdots, n \tag{10.56}$$

Remark 10.24 *In the case $n = 2$, this can be done always. However, in general, for $n \geq 3$, the system to determine the invariants is over-determined, thus has no solution.*

Proposition 10.8 *Assume that (10.53) is endowed with a coordinate of Riemann invariants $c(u) = (c_1(u), \cdots, c_n(u))$. Then*

(1) Its canonical form is given by (10.56), which is diagonal system.

(2) For any i, $i = 1, \cdots, n$, $c_i(u)$ is constant along an i-th characteristic associated with any smooth solution.

In particular, for any smooth solution $u(x,t)$

$$||c(u(\cdot,t))||_{L^\infty} = ||c(u_0)||_{L^\infty} \tag{10.57}$$

In the rest of this section, we always assume that (10.53) is endowed with a coordinate of Riemann invariants $c(u) = (c_1(u), \cdots, c_n(u))$, which can be normalized so that

$$\nabla c_i(u) \cdot r_j(u) = \delta_{ij} \tag{10.58}$$

Proposition 10.9 *Assume that (10.53) is endowed with a coordinate system of Riemann invariants such that (10.58) hold. Then*

$$(i) \qquad [r_j, r_k] = \nabla r_j \cdot r_k - \nabla r_k \cdot r_j = 0 \qquad \forall j, k \tag{10.59}$$

$$(ii) \qquad r_j^t \nabla^2 c_i r_k = -\nabla c_i \cdot \nabla r_j\, r_k = 0 \qquad i \neq j \neq k \neq i \tag{10.60}$$

$$(iii) \qquad \frac{\partial g_{jk}}{\partial c_i} = \frac{\partial g_{ji}}{\partial c_k} \qquad i \neq j \neq k \neq i \tag{10.61}$$

$$g_{kj} = \frac{1}{\lambda_k - \lambda_j}\frac{\partial \lambda_k}{\partial c_j} \tag{10.62}$$

Proof of Proposition 10.9: Recall that $u \mapsto c(u)$ is differomorphism, and

$$\frac{Du}{Dc}\frac{Dc}{Du} = I \Leftrightarrow \frac{Dc}{Du}\frac{Du}{Dc} = I$$

Then it follows from (10.58),

$$\frac{Du}{Dc} = R(u) = (r_1(u), \cdots, r_n(u)), \quad \frac{Dc}{Du} \equiv L(u)$$

i.e.

$$\frac{\partial u}{\partial c_i} = r_i(u), \quad r_i(u) = r_i(u)$$

Thus for any smooth function ϕ,

$$\frac{\partial \phi}{\partial c_i} = \nabla_u \phi \cdot r_i(u) = \nabla_u \phi \cdot r_i(u) \tag{10.63}$$

<u>Step 1</u>:

$$0 = \nabla(\nabla c_i(u) \cdot r_j(u)) r_k = r_j^t \, \nabla^2 \, c_i \, r_k + \nabla c_i \cdot \nabla r_j \, r_k$$

so

$$\nabla c_i \nabla r_j \, r_k = -r_j^t \, \nabla^2 \, c_i \, r_k \qquad \forall \, i, j, k = 1, \cdots, n \tag{10.64}$$

$$\nabla c_i \nabla r_k \, r_j = -r_k^t \, \nabla^2 \, c_i \, r_j \qquad \forall \, i, j, k = 1, \cdots, n$$

$$\nabla c_i [r_j, r_k] = 0 \qquad \Leftrightarrow \qquad [r_j, r_k] = 0$$

since it is true for all i, $\Rightarrow$

By (10.63), this is equivalently

$$\frac{\partial r_j}{\partial c_k} = \frac{\partial r_k}{\partial c_j}$$

<u>Step 2:</u>

$$\begin{aligned} A r_j &= \lambda_j \, r_j \\ \nabla(A r_j) r_k &= \nabla(\lambda r_j) r_k = \nabla \lambda_j r_k r_j + \lambda_j \nabla r_j r_k \\ A r_k &= \lambda_k \, r_k^j \\ \nabla(A r_k) r_j &= \nabla(\lambda_k \, r_k) r_j \\ r_j^t \, \nabla A r_k + A \nabla r_j \, r_k &= \nabla \lambda_j \, r_k \, r_j + \lambda_j \nabla r_j \, r_k \\ r_k^t \, \nabla A r_j + A \nabla r_k \, r_j &= \nabla \lambda_k \, r_j \, r_k + \lambda_k \nabla r_k \, r_j \end{aligned}$$

Since $A = \nabla F$, so ∇A is symmetric. Taking the difference, we have

$$A[r_j, r_k] = (\nabla \lambda_j \, r_k) r_j - (\nabla \lambda_k \, r_j) r_k + \lambda_j \, \nabla r_j \, r_k - \lambda_k \, \nabla r_k \, r_j$$

$$\begin{aligned} (\nabla \lambda_j \, r_k) r_j - (\nabla \lambda_k \, r_j) r_k &= \lambda_k \, \nabla r_k \, r_j - \lambda_j \, \nabla r_j \, r_k \\ &= (\lambda_k - \lambda_j) \nabla r_j \, r_k \end{aligned} \tag{10.65}$$

This implies that $\nabla r_j \, r_k$ is a linear combination of r_j and r_k. Now for $i \neq j$, $i \neq k$, $j \neq k$

$$\begin{aligned} \nabla c_i \, \nabla r_j \, r_k &= \frac{\nabla \lambda_j \, r_k}{\lambda_k - \lambda_j} \nabla c_i \, r_j - \frac{\nabla \lambda_k \, r_j}{\lambda_k - \lambda_j} \nabla c_i \, r_k \\ &= 0 \end{aligned} \tag{10.66}$$

Then (10.60) follows from (10.64) and (10.66).

<u>Step 3</u>: By (10.65),

$$\frac{\partial r_j}{\partial c_k} = \frac{\frac{\partial \lambda_j}{\partial c_k}}{\lambda_k - \lambda_j} r_j - \frac{\frac{\partial \lambda_k}{\partial c_j}}{\lambda_k - \lambda_j} r_k$$

i.e.

$$-\frac{\partial r_j}{\partial c_k} = g_{jk} \, r_j + g_{kj} \, r_k, \qquad j, k = 1, \cdots, n, \quad j \neq k \tag{10.67}$$

Differentiate the equality with respect to c_i,

$$-\frac{\partial^2 r_j}{\partial c_k\,\partial c_i} = \frac{\partial g_{jk}}{\partial c_i} r_j + g_{jk}\frac{\partial r_j}{\partial c_i} + g_{kj}\frac{\partial r_k}{\partial c_i} + \frac{\partial g_{kj}}{\partial c_i} r_k$$

Substitute (10.67) into this formula,

$$-\frac{\partial^2 r_j}{\partial c_k\,\partial c_i} = \frac{\partial g_{jk}}{\partial c_i} r_j - g_{jk}(g_{ji}\,r_j + g_{ij}\,r_i) - g_{kj}(g_{ki}\,r_k + g_{ik}\,r_i) + \frac{\partial g_{kj}}{\partial c_i} r_k$$

By the symmetry of i and k,

$$-\frac{\partial^2 r_j}{\partial c_i\,\partial c_k} = \frac{\partial g_{ji}}{\partial c_k} r_j - g_{ji}(g_{jk}\,r_j + g_{kj}\,r_k) - g_{ij}(g_{ik}\,r_i + g_{ki}\,r_k) + \frac{\partial g_{ij}}{\partial c_k} r_i$$

This implies

$$\left(\frac{\partial g_{jk}}{\partial c_i} - \frac{\partial g_{ji}}{\partial c_k}\right) r_j + r_k(\qquad\qquad\qquad) + r_i(\qquad\qquad\qquad) = 0$$

so

$$\frac{\partial g_{jk}}{\partial c_i} = \frac{\partial g_{ji}}{\partial c_k}$$

Theorem 10.11 *Assume that*

(i) (10.53) is endowed with a coordinate system of Riemann invariants $c(u) = (c_1(u), \cdots, c_n(u))$.

(ii) (10.53) is strictly hyperbolic.

(iii) $\exists\, i \in \{1, \cdots, n\}$ *such that the i-th family is genuinely nonlinear*

$$\nabla\lambda_i\, r_i \neq 0 \qquad \left(\frac{\partial \lambda_i}{\partial c_i} \neq 0\right)$$

(iv) $u_0 \in H^s_{ul}(\mathbb{R}^1)$, $s \geq 3$ *and* $\exists\, \xi_0 \in \mathbb{R}^1$ *such that*

$$\frac{d\, c_i(u_0(\xi))}{d\xi}\frac{\partial \lambda_i}{\partial c_i} < 0, \qquad \frac{\partial \lambda_i}{c_i} = \nabla\lambda_i \cdot r_i(k) \tag{10.68}$$

Then smooth solution forms a shock in finite time.

Proof of Theorem 10.11:

Step 1: By (10.57) in Proposition 10.8, $||c(u(\cdot,t))||_{L^\infty} = ||c(u_0)||_{L^\infty}$, so there are no shell singularities.

Step 2: To estimate $\partial_\xi u$. Set

$$\partial_\xi u = \sum_{i=1}^{n} w_i\, r_i, \qquad w_i = l_i \cdot \partial_\xi u = \nabla c_i(u)\, \partial_\xi u \tag{10.69}$$

so

$$w_i = \partial_\xi\, c_i \tag{10.70}$$

$$\frac{d}{dt} w_i = \partial_t\, w_i + \lambda_i\, \partial_\xi\, w_i = \sum \gamma_{ijk}\, w_i\, w_j \tag{10.71}$$

and

$$\begin{aligned}\gamma_{ijk} &= -\frac{1}{2}(\lambda_j - \lambda_k) l_i [r_j, r_k] - (\nabla\lambda_i \cdot r_j)\delta_{ik} \\ &= -\frac{\partial \lambda_i}{\partial c_j}\delta_{ik}\end{aligned}$$

$$\begin{aligned}\frac{d}{dt} w_i &= \sum_{j,k} \left(-\frac{\partial \lambda_i}{\partial c_j}\delta_{ik} \right) w_k\, w_j \\ &= \sum_j \left(-\frac{\partial \lambda_i}{\partial c_j} w_i\, w_j \right) \\ &= -\frac{\partial \lambda_i}{\partial c_i} w_i^2 - \left(\sum_{j\neq i} \frac{\partial \lambda_i}{\partial c_j} w_j \right) w_i\end{aligned} \tag{10.72}$$

<u>Step 3</u>: Find an integration factor for (10.72)

$$\begin{aligned}&\frac{d}{dt}\Phi(u) \\ =\ & \Phi'(u)\frac{du}{dt} \qquad\qquad (\text{In fact}, \Phi'(u) = \nabla\Phi(u)) \\ =\ & \Phi'(u)\left(\frac{\partial u}{\partial t} + \lambda_i\, \partial_\xi u \right)\end{aligned}$$

$$\partial_t u = -A \partial_\xi u = -A \sum_j w_j\, r_j = -\sum_j w_j\, \lambda_j\, r_j$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\Phi(u) &= \Phi'(u)\left(-\sum_j \lambda_j\, w_j\, r_j + \lambda_i \sum_j w_j\, r_j \right) \\ &= \Phi'(u) \sum_{j\neq i} (\lambda_i - \lambda_j) w_j\, r_j\end{aligned}$$

Thus for any smooth function $\Phi(u)$,

$$\begin{aligned}&\frac{d}{dt}\left(e^{\Phi(u)} w_i \right) = \frac{d}{dt} e^{\Phi(u)} w_i + e^{\Phi(u)} \frac{d}{dt} w_i \\ =\ & e^{\Phi(u)}\frac{d}{dt} w_i + e^{\Phi(u)}\Phi'(u) \sum_{j\neq i} (\lambda_i - \lambda_j) w_j\, r_j\, w_i \\ =\ & e^{\Phi(u)} \left\{ -\sum \frac{\partial \lambda_i}{\partial c_j} w_i\, w_j + \nabla\Phi(u) \sum_{j\neq i} (\lambda_i - \lambda_j) w_i\, w_j\, r_j \right\} \\ =\ & e^{\Phi(u)} \left\{ -\frac{\partial \lambda_i}{\partial c_i} w_i^2 - \sum_{j\neq i} \left(\frac{\partial \lambda_i}{\partial c_j} - \nabla\Phi(u) r_j (\lambda_i - \lambda_j) \right) w_i\, w_j \right\} \\ =\ & e^{\Phi(u)} \left\{ -\frac{\partial \lambda_i}{\partial c_i} w_i^2 - \sum_{j\neq i} \left(\frac{\partial \lambda_i}{\partial c_j} - \frac{\partial \Phi}{\partial c_j} (\lambda_i - \lambda_j) \right) w_i\, w_j \right\}\end{aligned}$$

Claim: One can choose an integral factor $\Phi(u)$ such that

$$\frac{\partial \Phi}{\partial c_j} = \frac{\frac{\partial \lambda_i}{\partial c_j}}{\lambda_i - \lambda_j} \qquad j \neq i \tag{10.73}$$

Assume that the claim (10.73) holds

$$\frac{d}{dt}\left(e^{\Phi(u)} w_i\right) = -\frac{\partial \lambda_i}{\partial c_i} e^{-\Phi(u)} \left(e^{\Phi(u)} w_i\right)^2$$

Claim is followed from (10.61) and (10.62).

§10.4.4 Formation of Singularities for the Compressible Euler System

Consider

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = 0 \\ \partial(\rho S) + \operatorname{div}(\rho S u) = 0 \end{cases} \tag{10.74}$$

We only treat the polytropic gas,

$$p = A\rho^\gamma\, e^s \qquad\qquad A > 0 \quad \text{constants}, \quad \gamma > 1$$

ρ: density, $u \in \mathbb{R}^3$: velocity, p: pressure, S: entropy, $x \in \mathbb{R}^3$, with initial data:

$$(\rho, u, S)(x, t = 0) = (\rho_0, u_0, S_0)(x) \tag{10.75}$$

with

$$(\rho_0, u_0, S_0)(x) = (\bar{\rho}, 0, \bar{S}), \qquad x \geq R$$

$\bar{\rho} > 0$, $\bar{S}$ are constants, $\rho_0(x) \geq 0$, $\forall\ x \in \mathbb{R}^3$.

Recall that for any $w \in S^2$, the characteristic speeds for (10.74) are given by $u \cdot w$, $u \cdot w \pm c$,

$$c^2 = p_\rho = \frac{\partial p}{\partial \rho}$$

Define

$$\bar{c}^2 = \frac{\partial p}{\partial \rho}(\bar{S}, \bar{\rho}) = A\gamma \bar{\rho}^{\gamma - 1}\, e^{\bar{S}}$$

Define

$$D(t) = \{x \in \mathbb{R}^3 | \quad |x| > R + \bar{c}t\}$$

Then

Proposition 10.10 *Let (ρ, u, S) be the C^1-solution to the Cauchy problem (10.74) and (10.75). Then*

$$(\rho, u, S)(x, t) = (\bar{\rho}, 0, \bar{S}) \qquad in \qquad D(t)$$

Proof of Proposition 10.10: Since the maximal speed of nonconstant state is $\bar{c}$, therefore, the conclusion follows by local energy principle.

First result concerns the “blow-up” of solution, whose initial radial momentum is “large” enough.

Define

$$m(t) = \int_{\mathbb{R}^3} (\rho(x, t) - \bar{\rho}) dx$$

it is called excessive mass.

$$\begin{aligned} \eta(t) &= \int_{\mathbb{R}^3} \left(\rho(x,t)e^{s/\gamma} - \bar{\rho}e^{\bar{s}/\gamma}\right) dx \\ F(t) &= \int_{\mathbb{R}^3} (x\cdot u)\,\rho(x,t)dx \end{aligned}$$

Theorem 10.12 *Assume that (ρ, u, S) be a C^1-smooth solutions to the Cauchy problem (10.74) and (10.75) on $\mathbb{R}^3 \times [0,T]$, $T > 0$. Furthermore,*

$$\begin{cases} m(0) \geq 0, \quad \eta(0) \geq 0, \\ F(0) > \dfrac{16}{3R^4}\pi\,\bar{c}\cdot \max \rho_0(x) \end{cases} \tag{10.76}$$

Then the life span of the solution is finite.

Proof of Theorem 10.12:

Step 1: $m(t) = m(0)$, $\eta(t) = \eta(0)$.

$$\begin{aligned} \frac{d}{dt}m(t) &= \int_{\mathbb{R}^3} \frac{\partial}{\partial t}(\rho(x,t) - \bar{\rho})dx \;=\; -\int_{\mathbb{R}^3} \operatorname{div}(\rho u)dx \\ &= -\int_{D(t)^c} \operatorname{div}(\rho u)dx \;=\; 0 \end{aligned}$$

Since $u|_{\partial D(t)^c} = 0$.

$$\begin{aligned} \frac{d}{dt}\eta(t) &= \int_{\mathbb{R}^3} \frac{\partial}{\partial t}\left(\rho e^{s/\gamma}\right) dx \;=\; \int_{\mathbb{R}^3} \frac{\partial}{\partial t}\rho e^{s/\gamma} + \rho\frac{1}{\gamma}\partial_t S\, e^{s/\gamma} dx \\ &= -\int_{\mathbb{R}^3} \operatorname{div}(\rho u)e^{s/\gamma} + \frac{\rho}{\gamma} u\cdot \nabla S\, e^{s/\gamma} dx \\ &= -\int_{\mathbb{R}^3} \operatorname{div}(\rho u \cdot e^{s/\gamma})dx \\ &= 0 \end{aligned}$$

Step 2:

$$\begin{aligned} \frac{d}{dt}F(t) &= \int_{\mathbb{R}^3} \frac{\partial}{\partial t}((x\cdot u)\rho)dx \\ &= \int_{\mathbb{R}^3} x\cdot(\rho u)_t dx \;=\; -\int_{\mathbb{R}^3} x(\operatorname{div}(\rho u\otimes u) + \nabla p)dx, \\ & -\int_{\mathbb{R}^3} x\cdot \operatorname{div}(\rho u \otimes u)dx \\ &= -\sum_{i=1}^{3}\int_{\mathbb{R}^3} x_i \left(\sum_{j=1}^{3} \partial_{x_j}(\rho u_j\, u_i)\right) dx \\ &= -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3} x_i\, \partial_{x_j}(\rho u_j\, u_i)dx \;=\; \sum_{i,j=1}^{3}\int_{\mathbb{R}^3} \frac{\partial x_i}{\partial x_j}\rho u_i\, u_j dx \\ &= \int_{\mathbb{R}^3} \rho|u|^2 dx, \\ & -\int_{\mathbb{R}^3} x\cdot\nabla p\, dx \;=\; -\int_{\mathbb{R}^3} x\cdot\nabla(p-\bar{p})dx \\ &= \int_{\mathbb{R}^3} (\operatorname{div} x)(p-\bar{p})dx = 3\int_{\mathbb{R}^3}(p-\bar{p})dx \end{aligned}$$

$$\frac{d}{dt}F(t) = \int_{\mathbb{R}^3} p|u|^2 dx + 3\int_{\mathbb{R}^3}(p-\bar{p})dx \tag{10.77}$$

Step 3: Set $B(t) = D^c(t)$.

$$\begin{aligned}
\int_{B(t)} p(x,t)dx &= \int_{B(t)} A\rho^\gamma\, e^S dx \;=\; A\int_{B(t)} \left(\rho e^{S/\gamma}\right)^\gamma\, dt \\
&\geq A\left(\int_{B(t)} 1\, dx\right)^{1-\gamma}\left(\int_{B(t)} \rho e^{S/\gamma}\, dx\right)^\gamma \\
&= A(\mathrm{vol}\,(B(t)))^{1-\gamma}\left(\int_{B(t)} \rho e^{S/\nu} - \bar{\rho}e^{\bar{S}/\nu} dx + \int_{B(t)} \bar{\rho}e^{\bar{S}/\nu} dx\right)^\gamma \\
&= A(\mathrm{vol}(B(t)))^{1-\gamma}\left(\eta(t) + \int_{B(t)} \bar{\rho}e^{\bar{S}/\gamma} dx\right)^\gamma \\
&= A(\mathrm{vol}(B(t)))^{1-\gamma}\left(\eta(0) + \bar{\rho}e^{\bar{S}/\gamma}\mathrm{vol}(B(t))\right)^\gamma \\
&\geq A(\mathrm{vol}(B(t)))^{1-\gamma}\left(\bar{\rho}e^{\bar{S}/\gamma}\ \mathrm{vol}(B(t))\right)^\gamma \\
&= A\bar{\rho}^\gamma\, e^{\bar{S}}\ \mathrm{vol}(B(t)) \;=\; \bar{p}\ \mathrm{vol}(B(t)) \;=\; \int_{B(t)} \bar{p}\, dx
\end{aligned}$$

so

$$\int_{B(t)} (p-\bar{p})dx \geq 0$$

i.e.

$$\int_{\mathbb{R}^3} (p-\bar{p})dx \geq 0 \tag{10.78}$$

Combining (10.77) and (10.78), which yields

$$\frac{d}{dt}F(t) \geq \int_{\mathbb{R}^3} \rho|u|^2 dx \tag{10.79}$$

Step 4:

$$\begin{aligned}
F^2(t) &= \left(\int_{\mathbb{R}^3} (x\cdot u)\,\rho(x,t)dx\right)^2 \\
&= \left(\int_{B(t)} (x\cdot u)\,\rho(x,t)dx\right)^2 \\
&\leq \left(\int_{B(t)} \rho|x|^2 dx\right)\left(\int_{B(t)} \rho|u|^2 dx\right) \\
&= \left(\int_{B(t)} \rho|x|^2 dx\right)\left(\int_{\mathbb{R}^3} \rho|u|^2 dx\right)
\end{aligned}$$

This, together with (10.79), implies,

$$\frac{d}{dt}F(t) \geq \left(\int_{B(t)} \rho|x|^2\, dx\right)^{-1} F^2(t) \tag{10.80}$$

$$
\begin{aligned}
\int_{B(t)} \rho|x|^2\,dx &\le (R+\bar{c}t)^2 \int_{B(t)} \rho\,dx \\
&= (R+\bar{c}t)^2 \left(\int_{B(t)} (\rho-\bar{\rho})dx + \bar{\rho}\,\mathrm{vol}\,(B(t)) \right) \\
&= (R+\bar{c}t)^2 (m(0) + \bar{\rho}\,\mathrm{vol}\,(B(t))) \\
&= (R+\bar{c}t)^2 \left(\int_{B(t)} \rho_0(x)dx - \int_{B(t)} \bar{\rho}\,dx + \bar{\rho}\,\mathrm{vol}\,(B(t)) \right) \\
&\le (R+\bar{c}t)^2 \ \max\ \rho_0\,\mathrm{vol}\,(B(t)) \\
&\le \frac{4}{3}\Pi(R+\bar{c}t)^5 \max\ \rho_0
\end{aligned}
$$

i.e.

$$
\left(\int_{B(t)} \rho|x|^2 dx \right)^{-1} \ge \left(\frac{4}{3}\Pi(R+\bar{c}t)^5 \max\ \rho_0 \right)^{-1} \tag{10.81}
$$

(10.80) and (10.81) will yield,

$$
\frac{d}{dt}F(t) \ge \left(\frac{4}{3}\Pi(R+\bar{c}t)^5 \max\ \rho_0 \right)^{-1} F^2(t)
$$

so

$$
\int_0^T \frac{F'(t)}{F^2(t)}dt \ge \left(\frac{4}{3}\Pi\ \max\ \rho_0 \right)^{-1} \int_0^T (R+\bar{c}t)^{-5}dt
$$

and

$$
F^{-1}(0) \ge F^{-1}(0) - F^{-1}(t) \ge \left(\frac{16\bar{c}}{3}\Pi\ \max\ \rho_0 \right)^{-1} (R^{-4} - (R+cT)^{-4})
$$

$$
R^{-4} - (R+\bar{c}T)^{-4} \le \frac{\frac{16\bar{c}}{3}\Pi\ \max\ \rho_0}{F(0)}
$$

$$
0 < R^{-4} - \frac{\frac{16\bar{c}}{3}\Pi\ \max\ \rho_0}{F(0)} \le (R+cT)^{-4}
$$

The second result concerns the singularity formation without the "Largeness" requirement.

$$
q_0(\nu) = \int_{|x|>\nu} |x|^{-1}\,(|x|-\nu)^2\,(\rho_0(x)-\bar{\rho})dx \tag{10.82}
$$

$$
q_1(\nu) = \int_{|x|>\nu} |x|^{-3}\,(|x|^2-\nu^2)(x\cdot u_0)\rho_0\,dx \tag{10.83}
$$

Theorem 10.13 *Suppose that $\exists\, R_0$, and R such that*

$$
(i) \qquad q_0(\nu) > 0, \quad q_1(\nu) \ge 0, \quad R_0 < \nu < R \tag{10.84}
$$

$$
(ii) \qquad S_0(x) \ge \bar{S} \tag{10.85}
$$

Then life span of any C^1-smooth solution must be finite.

Remark 10.25 *The argument to prove Theorem 10.13 depends crucially on the Riemann function of $\Box$ operators.*

We need an elementary lemma.

Lemma 10.4 *Assume that there exist a positive constants C, A, k and $k_1 (0 < k_1 \leq \frac{k}{2})$, let $F(t)$ be any C^2-smooth function with*

$$F(0) = F'(0) = 0$$

such that

$$F''(t) \geq C\left[(t+k)^3 \log\left(\frac{t+k}{k}\right)\right]^{-1} F^2(t), \quad t \geq k_1 \tag{10.86}$$

$$F''(t) > 0 \qquad \forall\, t > 0 \tag{10.87}$$

$$F'(t) \geq \frac{1}{2} a \log\left(\frac{t+k}{k}\right), \qquad t \geq 0 \tag{10.88}$$

$$F(t) \geq ca(t+k)\log\left(\frac{t+k}{k}\right), \qquad t \geq k_1 \tag{10.89}$$

Then the life span of $F(t)$ is finite.

Proof of Lemma 10.4:
Step 1: (10.86) and (10.89) implies

$$\begin{aligned} F''(t) &\geq c\left[(t+k)^3 \log\left(\frac{t+k}{k}\right)\right]^{-1} c^2\, a^2 (t+k)^2 \log^2 \frac{t+k}{k}, \quad t \geq k_1 \\ &\geq c^3\, a^2 (t+k)^{-1} \log \frac{t+k}{k} \end{aligned}$$

Consequently,

$$F(t) \geq c^3\, a^2 (t+k)\left(\log \frac{t+k}{k}\right)^2, \qquad t \geq k_2 = 2k > k_1 \tag{10.90}$$

Substitute this into (10.86),

$$\begin{aligned} F''(t) &\geq c\left[(t+k)^3 \log\left(\frac{t+k}{k}\right)\right]^{-1} F^2(t) \\ &\geq c^4\, a^2 (t+k)^{-2} \log\left(\frac{t+k}{k}\right) F(t) \end{aligned}$$

Set $\mu(t) = c^4\, a^2 (t+k)^{-2} \log \frac{t+k}{k}$,

$$F''(t) \geq \mu(t)\, F(t) \tag{10.91}$$

Multiply $F'(t)$ on both sides of (10.91),

$$F'(t)\, F''(t) \geq \mu\, F'(t)\, F(t), \qquad t \geq k_2 \tag{10.92}$$

Now for any $k_3 \geq k_2$, $t \geq k_3$,

$$\begin{aligned}
\int_{k_3}^{t} F^{''}(t)\, F'(t)dt &\geq \int_{k_3}^{t} \mu\, F'(t)\, F\, ds \\
\frac{1}{2}(F'(t))^2 - \frac{1}{2}(F'(k_3))^2 &\geq \frac{1}{2}\int_{k_3}^{t} \mu \frac{dF^2}{ds} \\
&= \frac{1}{2}\mu(t)\, F^2(t) - \frac{\mu(k_3)}{2} F^2(k_3) - \frac{1}{2}\int_{k_3}^{t} \mu'\, F^2\, ds \\
(F'(t))^2 &\geq (F'(k_3))^2 + \mu(t)\, F^2(t) - \mu(k_3) F^2(k_3) - \int_{k_3}^{t} \mu'\, F^2 ds
\end{aligned} \tag{10.93}$$

Since

$$\begin{aligned}
\mu'(t) &= c^4\, a^2 \left(-2(t+k)^{-3} \log \frac{t+k}{k} + (t+k)^{-2} \frac{1}{t+k} \right) \\
&= c^4\, a^2 \left(1 - 2\log \frac{k+t}{k} \right) (t+k)^{-3} < 0 \qquad (t \geq 2k)
\end{aligned}$$

Therefore,

$$(F'(t))^2 \geq \mu(t)\, F^2(t) + (F'(k_3))^2 - \mu(k_3)\, F^2(k_3) \tag{10.94}$$

<u>Step 2</u>: Since $F'(t)$ is increasing, due to (10.87), for $0 < t_1 < t_2$.

$$F'(t_1) \leq \frac{F(t_2) - F(t_1)}{t_2 - t_1} \leq F'(t_2) \tag{10.95}$$

$$F(k_3) \leq F'(k_3) k_3 \tag{10.96}$$

Now, we choose k_3 such that

$$1 \leq k_3^2\, \mu(k_3) = k_3^2\, c^4\, a^2 (k_3 + k)^{-2} \log\left(\frac{k_3 + k}{k}\right) \tag{10.97}$$

it suffices to choose

$$k_3 \sim 0\left(e^{\frac{1}{c^4\, a^2}}\right) \tag{10.98}$$

Then

$$\begin{aligned}
(F'(t))^2 &\geq (F'(k_3))^2 + \int_{k_3}^{t} \mu(s)(F^2(s))' ds \\
&\geq (F'(k_3))^2 + (k_3^2\, \mu(k_3))^{-1} \int_{k_3}^{t} \mu(s)(F^2(s))' ds \\
&= (F'(k_3))^2 + (k_3^2\, \mu(k_3))^{-1} \left[\mu(t)\, F^2(t) - \mu(k_3)\, F^2(k_3)\right] - (k_3^2\, \mu(k_3))^{-1} \int_{k_3}^{t} \mu'(s)\, F^2(s)\, ds \\
&= \frac{\mu(t)}{\mu(k_3)\, k_3^2} F^2(t) + (F'(k_3))^2 - \frac{1}{k_3^2} F^2(k_3) \\
&\geq \frac{\mu(t)}{k_3^2\, \mu(k_3)} F^2(s) \\
\frac{\mu(t)}{k_3^2\, \mu(k_3)} &= \frac{c^4\, a^2}{k_3^2\, \mu(k_3)} (t+k)^{-2}\, \log\left(\frac{t+k}{k}\right)
\end{aligned}$$

Immediately,

$$F'(t) \geq c^2\, a(t+k)^{-1} \left(\log\left(\frac{t+k}{k}\right)\right)^{1/2} F(t) \qquad (\text{take } k_3^2\, \mu(k_3) = 1) \qquad t \geq k_3 \tag{10.99}$$

so

$$\begin{aligned} \log \frac{F(t)}{F(k_3)} &\geq ca \left[\left(\log \frac{t+k}{k}\right)^{3/2} - \left(\log \frac{k_3+k}{k}\right)^{3/2} \right] \\ &\geq ca \left(\log \frac{t+k}{k} - \log \frac{k_3+k}{k}\right)^{3/2} \\ &\geq ca \left(\log \frac{t+k}{k_3+k}\right)^{3/2} \end{aligned}$$

Now, let choose $c > 0$ large enough so that $t \geq k_4 = \tilde{c}\, k_3^2$.

$$\log \frac{F(t)}{F(k_3)} \geq 8 \log \frac{t+k}{k} \tag{10.100}$$

Note that (10.100) requires that

$$ca \left(\log \frac{\tilde{c}\, k_3^2 + k}{k_3 + k}\right)^{1/2} \geq c_0$$

This can be guaranteed by (10.98). Thus

$$\begin{aligned} F(t) &\geq F(k_3) \left(\frac{t+k}{k}\right)^8 \qquad t \geq k_4 \\ &\geq ca^2(k_3+k) \left(\log \frac{k_3+k}{k}\right)^2 k^{-8}(t+k)^8 \\ &\geq ca^2(t+k)^8 \qquad\qquad t \geq k_4 \end{aligned}$$

i.e.

$$F(t) \geq ca^2(t+k)^8 \qquad\qquad t \geq k_4 \tag{10.101}$$

It follows from (10.101) and (10.86),

$$F''(t) \geq ca(F(t))^{3/2} \qquad\qquad t \geq k_4 \tag{10.102}$$

Multiply $F'(t)$ on the both sides of (10.102), then

$$(F'(t))^2 \geq ca \left[(F(t))^{5/2} - (F(k_4))^{5/2}\right]$$

On the other hand,

$$F(t) \geq F(k_4) + F'(k_4)(t - k_4)$$

$$F(k_4) \leq k_4\, F'(k_4)$$

$$F(t) \geq F'(k_4)(t-k_4) \geq F'(k_4)\frac{t-k_4}{k_4}$$

$$
\begin{aligned}
F^{5/2}(t) - F^{5/2}(k_4) &= \frac{1}{2}F^{5/2}(t) + \frac{1}{2}F^{5/2}(t) - F^{5/2}(k_4) \\
&\geq \frac{1}{2}F^{5/2}(t) + (F'(k_4))^{5/2}\left[\left(\frac{t-k_4}{k_4}\right)^{5/2}\frac{1}{2} - 1\right]
\end{aligned}
$$

Now if $k_5 \geq 3k_4$, then if $t \geq k_5$,

$$
F'(t) \geq ca^{\frac{1}{2}}\,F^{\frac{5}{4}}(t)
$$

Proof of Theorem 10.13: In the following, we will assume $\nu = 2$, the general case can be handled by obvious modifications, so let (ρ, u, s) be any C^1-smooth solution to the Cauchy problem. We will construct a functional $F(t)$ in terms of (ρ, u, s) so that $F(t)$ satisfies all the conditions in Lemma 10.4.

Step 1: Construct a 2-variable function as

$$
Q(\nu, t) = \int_{|x|>\nu} w(x,\nu)(\rho(x,t) - \bar{\rho})dx \tag{10.103}
$$

where

$$
w(x,\nu) = |x|^{-1}\,(|x| - \nu)^2 \tag{10.104}
$$

Note that

$$
\rho(x,t) = \bar{\rho} \qquad \text{for all} \quad |x| > R + \bar{c}t
$$

so $Q(\nu, t)$ is well-defined and at least C^1.

Step 2:

Claim: $Q \in C^2$, and

$$
\Box\, Q = \partial_t^2\, Q - \bar{c}^2\,\partial_\nu^2\, Q \geq G(\nu, t) \tag{10.105}
$$

$$
\begin{aligned}
G(\nu,t) &= \partial_\nu^2\,\tilde{G}(\nu,t) \\
\tilde{G}(\nu,t) &= \int_{|x|>\nu} w(x,\nu)(p - \bar{p} - \bar{c}^2(\rho - \bar{\rho}))dx
\end{aligned} \tag{10.106}
$$

Proof of the claim:

$$
\begin{aligned}
\partial_t\, Q(\nu,t) &= \int_{|x|>\nu} w(x,\nu)\,\partial_t\rho(x,t)dx \\
&= -\int_{|x|>\nu} w(x,\nu)\,\mathrm{div}\,(\rho u)dx \\
&= \int_{|x|>\nu} \nabla_x w(x,\nu)\cdot(\rho u)dx
\end{aligned}
$$

Thus $\partial_t\, Q(\nu, t)$ is a C^1-function,

$$
\begin{aligned}
\partial_t^2\, Q(\nu,t) &= \int_{|x|>\nu} \partial_t(\rho u)\cdot\nabla_x\, w(x,\nu)dx \\
&= -\int_{|x|>\nu} (\mathrm{div}\,(\rho u\otimes u) + \nabla p)\cdot\nabla_x\, w(x,\nu)dx \\
&= -\int_{|x|>\nu} \partial_{x_i} w\cdot\partial_{x_j}(\rho\, u_i\, u_j) + \partial_{x_j}(p - \bar{p})\,\partial_{x_i} w(x,\nu)dx
\end{aligned}
$$

Note that

$$\nabla w = |x|^{-3}\,(|x|^2-\nu^2)x \qquad \nabla w|_{|x|=\nu} = 0$$
$$\operatorname{supp} u \subset B_{R+\bar c t}, \qquad \operatorname{supp}(p-\bar p) \subset B_{R+\bar c t}$$

so

$$\partial_t^2\, Q(\nu,t) = \sum_{i,j=1}^{3} \int_{|x|>\nu} \frac{\partial^2 w}{\partial x_i\,\partial x_j} \rho\, u_i\, u_j\, dx + \int_{|x|>\nu} \Delta w (p-\bar p) dx \tag{10.107}$$

$$\partial_t^2\, Q = I_1(\nu,t) + I_2(\nu,t) \tag{10.108}$$

Note that

$$\frac{\partial^2\, w}{\partial x_i\,\partial x_j} = \frac{|x|^2-\nu^2}{|x|^3}\delta_{ij} - 3\frac{|x|^2-\nu^2}{|x|^5} x_i\, x_j + 2\frac{x_i\, x_j}{|x|^3}$$

$$\begin{aligned} I_1(\nu,t) &= \int_{|x|>\nu} \frac{|x|^2-\nu^2}{|x|^3}\rho |u|^2\, dx - 3\int_{|x|>\nu} \frac{|x|^2-\nu^2}{|x|^5}\rho (x\cdot u)^2 dx + 2\int_{|x|>\nu} \frac{\rho (x\cdot u)^2}{|x|^3} dx \\ &= 2\nu^2 \int_{|x|>\nu} \rho \frac{(x\cdot u)^2}{|x|^5} dx + \int_{|x|>\nu} \frac{|x|^2-\nu^2}{|x|^3}\rho |u|^2 dx - \int_{|x|>\nu} \frac{|x|^2-\nu^2}{|x|^3}\rho \left(\frac{x}{|x|} - u\right)^2 \end{aligned}$$

Therefore,

$$I_1(\nu,t) \geq 2\nu^2 \int_{|x|>\nu} \rho \frac{(x\cdot u)^2}{|x|^5} dx \tag{10.109}$$

Note that

$$\Delta w = \frac{2}{|x|}$$

Therefore,

$$I_2(\nu,t) = \int_{|x|>\nu} \Delta w (p-\bar p) dx = 2\int_{|x|>\nu} |x|^{-1} (p-\bar p) dx$$

On the other hand,

$$\partial_\nu\, w(x,\nu) = 2|x|^{-1}\,(\nu - |x|)$$
$$\partial_\nu^2\, w(x,\nu) = \frac{2}{|x|}$$
$$I_2(\nu,t) = \int_{|x|>\nu} \partial_\nu^2\, w(p-\bar p) dx \equiv \frac{\partial}{\partial\nu^2} \int_{|x|>\nu} w(p-\bar p)$$

$$\begin{aligned} \partial_t^2\, Q &= I_1(\nu,t) + I_2(\nu,t) \;\geq\; I_2(\nu,t) \\ &= \partial_\nu^2 \int_{|x|>\nu} w\cdot (p-\bar p) dx \\ &= \partial_\nu^2 \int_{|x|>\nu} w\cdot (p-\bar p - \bar c^2(\rho-\bar\rho)) dx + \partial_\nu^2 \int_{|x|>\nu} w\cdot \bar c^2 (\rho - \bar\rho) dx \\ &= G(\nu,t) + \bar c^2\, \partial_\nu^2\, Q(\nu,t) \end{aligned}$$

This verifies the claim.

Next, we check the initial condition for Q.

$$Q(\nu, t=0) = \int_{|x|>\nu} w(x,\nu)(\rho_0(x) - \bar{\rho})dx = q_0(\nu)$$

$$\partial_t\, Q(\nu, t=0) = \int_{|x|>\nu} \rho_0\, u_0 \cdot \nabla w\, dx = \int_{|x|>\nu} \rho_0\, u_0 \left(\frac{|x|^2 - \nu^2}{|x|^3} x \right) dx = q_1(\nu)$$

$$G(\nu, t) = \int_{|x|>\nu} 2|x|^{-1}\,(p - \bar{p} - \bar{c}^2(\rho - \bar{\rho}))dx \tag{10.110}$$

Thus, applying the one dimensional D' Alembertian formula for

$$\Box = \partial_t^2 - \bar{c}^2\,\partial_\nu^2$$

we obtain for $\nu > R_0 + \bar{c}t$.

$$Q(\nu, t) = Q_0(\nu, t) + \frac{1}{2\bar{c}} \int_0^t \int_{\nu-\bar{c}(t-\tau)}^{\nu+\bar{c}(t-\tau)} \Box\, Q(y,\tau)dy\, d\tau \tag{10.111}$$

$$Q_0(\nu, t) = \frac{1}{2}\left\{ q_0(\nu + \bar{c}t) + q_0(\nu - \bar{c}t) + \frac{1}{\bar{c}} \int_{\nu-\bar{c}t}^{\nu+\bar{c}t} q_1(y)dy \right\} \tag{10.112}$$

Then

$$Q(\nu, t) \geq Q_0(\nu, \tau) + \frac{1}{2\bar{c}} \int_0^t \int_{\nu-\bar{c}(t-\tau)}^{\nu+\bar{c}(t-\tau)} G(y,\tau)dy\, d\tau$$

Step 3: Set

$$\begin{aligned} F(t) &= \int_0^t (t-\tau) \int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} \nu^{-1}\, Q(\nu,\tau) dr\, d\tau \\ &= \int_0^t (t-\tau) \int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} \nu^{-1} \int_{\nu<|x|\leq R+\bar{c}\tau} w(x,\nu)(\rho,\bar{\rho})dx\, dr\, d\tau \end{aligned} \tag{10.113}$$

Obviously, $F(t) \in C^2$.

$$F'(t) = \int_0^t \int_{R_0+\bar{c}t}^{R+\bar{c}\tau} \nu^{-1} \int_{|x|>\nu} w(x,\nu)(\rho(x,\tau) - \bar{\rho})dx\, d\nu\, d\tau \tag{10.114}$$

Look at (10.85), $s_0(x) \geq \bar{s}$, $x \in \mathbb{R}^3$. For smooth solution

$$\partial_t s + u \cdot \nabla s = 0$$

Along particle path,

$$\frac{dx}{dt} = u$$

$s(x(t),t) = s_0(x(0))$, so

$$s(\nu, t) \geq \bar{s} \qquad \forall\, x, t$$

$$p(\rho, s) = A\rho^\nu\, e^s, \qquad \text{so} \qquad p(\rho, s) \geq p(\rho, \bar{s})$$

$$
\begin{aligned}
p-\bar{p}-\bar{c}^2(\rho-\bar{\rho}) &= Ae^{s}\,\rho^2 - A\bar{\rho}^2\,e^{\bar{s}} - 2A\bar{\rho}\,e^{\bar{s}}(\rho-\bar{\rho})\\
&\geq Ae^{\bar{s}}\,\rho^2 - 2A\bar{\rho}^2\,e^{\bar{s}} - 2A\bar{\rho}\,e^{\bar{s}}(\rho-\bar{\rho})\\
&= Ae^{\bar{s}}(\rho^2-\bar{\rho}^2-2\rho\bar{\rho}+2\bar{\rho}^2)\\
&= Ae^{\bar{s}}(\rho-\bar{\rho})^2
\end{aligned}
$$

$$
\begin{aligned}
G(\nu,t) &= \int_{|x|>\nu} |x|^{-1}(p-\bar{p}-\bar{c}^2(\rho-\bar{\rho}))dx\\
&\geq Ae^{\bar{s}}\int_{|x|>\nu} |x|^{-1}(\rho-\bar{\rho})^2 dx \;\geq\; 0
\end{aligned}
$$

so

$$Q(\nu,t) \geq Q_0(\nu,t) > 0$$

This implies

$$F(0)=F'(0)=0, \qquad \text{and} \qquad F'(t)>0, \quad \forall\, t>0$$

Therefore

$$F(t)>0 \qquad \text{for} \qquad t>0$$

Now let us compute $F^{''}(t)$,

$$
\begin{aligned}
F^{''}(t) &= \int_{R_0+\bar{c}t}^{R+\bar{c}t} \nu^{-1}\,Q(\nu,t)d\nu\\
&\geq \int_{R_0+\bar{c}t}^{R+\bar{c}t} \nu^{-1}\,Q_0(\nu,t)d\nu + \frac{1}{2\bar{c}}\int_{R_0+\bar{c}t}^{R+\bar{c}t}\int_0^t\int_{\nu-\bar{c}(t-\tau)}^{\nu+\bar{c}(t-\tau)} \nu^{-1}\,G(y,\tau)dy\,d\tau\,d\nu\\
&= J_1(t)+J_2(t)\\
J_1(t) &= \int_{R_0+\bar{c}t}^{R+\bar{c}t} \nu^{-1}\,Q_0(\nu,t)d\nu \;\geq\; \frac{1}{2}\int_{R_0+\bar{c}t}^{R+\bar{c}t} \nu^{-1}\,q_0(\nu-\bar{c}t)d\nu\\
&\geq \frac{1}{2}\int_{R_0+\bar{c}t}^{R+\bar{c}t} (R+\bar{c}t)^{-1}\,q_0(\nu-\bar{c}t)d\nu\\
&= \frac{B_0}{2}(R+\bar{c}t)^{-1}
\end{aligned}
$$

where

$$B_0 = \int_{R_0}^{R} q_0(\nu)d\nu > 0$$

<u>Estimate of J_2</u>:

$$J_2(t) = \frac{1}{2\bar{c}}\int_{R_0+\bar{c}t}^{R+\bar{c}t}\int_0^t\int_{\nu-\bar{c}(t-\tau)}^{\nu+\bar{c}(t-\tau)} \nu^{-1}\,G(y,\tau)dy\,d\tau\,d\nu$$

Note that

$$\operatorname{supp} G(y,\tau) = \{y \leq R+\bar{c}\tau\}$$

Then changing of order the integration in $J_2(t)$, we can get that for $t \geq R_1 = \dfrac{1}{2\bar{c}}(R-R_0) > 0$.

$$J_2(t) = \frac{1}{2\bar{c}}\int_0^t\int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} G(y,\tau)\int_{\max\{\bar{c}t+R_0,\,y-\bar{c}(t-\tau)\}}^{\min\{R+\bar{c}t,\,y+\bar{c}(t-\tau)\}} \nu^{-1}d\nu\,dy\,d\tau$$

$$
\begin{aligned}
\Delta_1 &= \int_{\max\{\bar{c}t+R_0,\,y-\bar{c}(t-\tau)\}}^{\min\{R+\bar{c}t,\,y+\bar{c}(t-\tau)\}} \nu^{-1}dy\\
&\geq (R+ct)^{-1}\,(\min\{R+\bar{c}t,\,y+\bar{c}(t-\tau)\} - \max\{R_0+\bar{c}t,\,y-\bar{c}(t-\tau)\})
\end{aligned}
$$

Note that $y \le R + \bar{c}\tau$, then

$$R + \bar{c}t = R + \bar{c}(t-\tau) + \bar{c}\tau \ge y + \bar{c}(t-\tau)$$

so

$$\min\{R + \bar{c}t, y + \bar{c}(t-\tau)\} = y + \bar{c}(t-\tau)$$

Thus,

$$\Delta_1 \ge (R + \bar{c}t)^{-1} \min\{2\bar{c}(t-\tau), y - \bar{c}\tau - R_0\}$$

<u>Case 1</u>: $\min\{2\bar{c}(t-\tau), y - \bar{c}\tau - R_0\} = y - \bar{c}\tau - R_0$

Since

$$\frac{\bar{c}(t-\tau)}{\bar{c}t + R} < 1, \qquad y - \bar{c}\tau - R_0 \le R - R_0$$

$$\begin{aligned} \Delta_1 &\ge (R + \bar{c}t)^{-1} \cdot 1 \cdot (y - \bar{c}\tau - R_0) \cdot 1 \\ &\ge \frac{\bar{c}(t-\tau)}{(R+\bar{c}t)^2} \frac{(y - \bar{c}\tau - R_0)^2}{R - R_0} \\ &= \frac{\bar{c}}{R - R_0} (R + \bar{c}t)^{-2} (t-\tau) (y - \bar{c}\tau - R_0)^2 \end{aligned}$$

<u>Case 2</u>: $\min\{2\bar{c}(t-\tau), y - \bar{c}\tau - R_0\} = 2\bar{c}(t-\tau)$

$$\begin{aligned} \Delta_1 &\ge (R + \bar{c}t)^{-1} 2\bar{c}(t-\tau) \\ &\ge 2\bar{c}(R + \bar{c}t)^{-1} \frac{R}{R + \bar{c}t} (t-\tau) \left(\frac{y - \bar{c}\tau - R_0}{R - R_0}\right)^2 \\ &= \frac{2R}{(R - R_0)^2} \bar{c}(R + \bar{c}t)^{-2} (t-\tau) (y - \bar{c}\tau - R_0)^2 \end{aligned}$$

In summary, we have shown that $\exists\, C_0(R_0, R) = C_0$, such that

$$\Delta_1 \ge C_0\, \bar{c}(R + \bar{c}t)^{-2} (t-\tau) (y - \bar{c}\tau - R_0)^2$$

Therefore, for $t \ge R_1$, we have

$$\begin{aligned} J_2(t) &\ge \frac{C_0\,\bar{c}}{2\bar{c}} \int_0^t \int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} (t-\tau)(R + \bar{c}t)^{-2} (y - \bar{c}\tau - R_0)^2\, G(y,\tau) dy\, d\tau \\ &= \frac{C_0}{2} (R + \bar{c}t)^{-2} \int_0^t \int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} (t-\tau) (y - \bar{c}\tau - R_0)^2\, G(y,\tau) dy\, d\tau \end{aligned}$$

Recall that

$$G(y,\tau) = \partial_y^2 \tilde{G}(y,\tau)$$

$$\tilde{G}(y,\tau) = \int_{|x|>y} w(x,y)(p - \bar{p} - \bar{c}^2(\rho - \bar{\rho}))dx$$

so

$$\begin{aligned} J_2(t) &\ge C_0(R + \bar{c}t)^{-2} \int_0^t \int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} (t-\tau)\, \tilde{G}(y,\tau) dy\, d\tau \\ &\ge C_0(R + \bar{c}t)^{-2} \left(\int_0^t \int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} \int_{|x|>y} (t-\tau)\, w(x,y)(\rho - \bar{\rho})^2 dx\, dy\, d\tau \right) \frac{\bar{c}^2}{\bar{\rho}} \\ &= C_0 \frac{\bar{c}^2}{\bar{\rho}} (R + \bar{c}t)^{-2} J_3(t) \end{aligned}$$

where

$$J_3(t) = \int_0^t \int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} \int_{|x|>y} (t-\tau)\, w(x,y)(\rho-\bar{\rho})^2 dx\, dy\, d\tau$$

$$\begin{aligned} F^2(t) &= \left\{ \int_0^t \int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} \int_{|x|>\nu} \nu^{-1}(t-\tau)\, w(x,\nu)(\rho-\bar{\rho}) dx\, d\nu\, d\tau \right\}^2 \\ &\leq \int_0^t \int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} \int_{|x|>\nu} (t-\tau)\, w(x,\nu)(\rho-\bar{\rho})^2 dx\, d\nu\, d\tau \\ &\quad \cdot \left(\int_0^t \int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} \int_{R+\bar{c}\tau>|x|>\nu} \nu^{-2}\, w(x,\nu)(t-\tau)\, dx\, d\nu\, d\tau \right) \\ &= J_3(t)\, J_4(t) \end{aligned}$$

$$\begin{aligned} J_4(t) &= \int_0^t \int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} \int_{R+\bar{c}\tau>|x|>y} y^{-2}\, w(x,y)(t-\tau)\, dx\, dy\, d\tau \\ &= 4\Pi \int_0^t (t-\tau) \int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} y^{-2} \int_y^{R+\bar{c}\tau} |x|^{-1}(|x|-y)^{-2}\, |x|^2\, d|x|\, dy\, d\tau \\ &\leq C_0 \int_0^t \int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} (t-\tau)\, y^{-2}\, (R+\bar{c}\tau)(R+\bar{c}\tau-y)^2\, dy\, d\tau \\ &\leq C_0\, \bar{c}^2 (R+\bar{c}t)\, \log \frac{R+\bar{c}t}{R} \end{aligned}$$

Therefore

$$\begin{aligned} J_3(t) &\geq (J_4(t))^{-1}\, F^2(t) \\ &\geq C_0\, \bar{c}^2 (R+\bar{c}t)^{-1} \left(\log \frac{R+\bar{c}t}{R} \right)^{-1} F^2(t) \end{aligned}$$

so

$$\begin{aligned} F^{''}(t) &\geq J_1(t) + J_2(t) \;>\; J_2(t) \;\geq\; C_0 \frac{\bar{c}^2}{\bar{\rho}} (R+\bar{c}t)^{-2}\, J_3(t) \\ &\geq C_0^2 \frac{\bar{c}^4}{\bar{\rho}} (R+\bar{c}t)^{-3} \left(\log \frac{R+\bar{c}t}{R} \right)^{-1} F^2(t) \end{aligned}$$

Since

$$\begin{aligned} J_2(t) &\geq 0 \qquad F^{''}(t) \geq J_1(t) \geq B_0 (R+\bar{c}t)^{-1} \quad \forall\, t>0 \\ F'(0) &= 0 \\ F'(t) &\geq \int_0^t J_1'(t) dt = \bar{c}^{-1}\, B_0\, \log \left(\frac{R+\bar{c}t}{R} \right) \qquad \forall\, t>0 \end{aligned}$$

$F(0) = 0$, so

$$F(t) = \int_0^t F'(\tau) d\tau \geq C_0\, \bar{c}^{-2}\, B_0 (R+\bar{c}t)\, \log \frac{R+\bar{c}t}{R}, \qquad \forall\, t > t_1$$