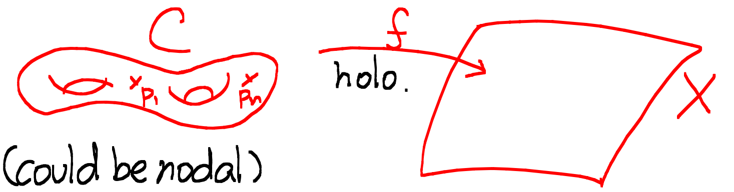
§ Gromov-Witten invariants

(ref: Cox-Katz, Ch7, GW inv.)

$\overline{\mathcal{M}}_{g,n}(X,\beta) \ni$ 

$g = g(C)$ $\qquad\qquad$ $\beta = f_*[C] \in H_2(X,\mathbb{Z})$

$n = \#$ marked pt. (distinct) on C $\qquad$ (topo. degree).

stable (i.e. $\#\operatorname{Aut}(C \xrightarrow{f} X) < \infty$)

(Alexeev) $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is a projective scheme $/\mathbb{C}$

Theorem: $\overline{\mathcal{M}}_{g,n}(X,\beta)$ alg. stack, proper$/\mathbb{C}$

$\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,\beta)$ smooth stack/orbifold

(expected) $\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n}(X,\beta) = (1-g)(\dim_{\mathbb{C}} X - 3) + \int_\beta c_1(X) + n$

"Count" $\overline{\mathcal{M}}_{g,n}(X,\beta)$ by imposing constraints.

(i) Passing thru. $Z_i \subset X$

(ii) Constraints on $(C, p_1, \dots, p_n) \in \overline{\mathcal{M}}_{g,n}$

((iii) Descendant constraints $\qquad$ $C \to L_i \supseteq T^*_{p_i}C$

$\psi_i = c_1(L_i) \in H^2(\overline{\mathcal{M}}_{g,n}(X,\beta))$

$$\begin{array}{ccccc} & & L_i & & T^*_{p_i}C \\ & & \downarrow & & \downarrow \\ \overline{\mathcal{M}}..(X,\beta) & \ni & C & \xrightarrow{f} & X \\ & & p_1,\dots,p_n & & \end{array}$$)

$$\begin{array}{ccccc} C \xrightarrow{f} X & \in & \overline{\mathcal{M}}_{g,n}(X,\beta) & \xrightarrow{P_i} & X \\ p_1,\dots,p_n & & F \Big\downarrow \text{forget} & & \rotatebox{90}{$\in$} \\ \downarrow & & & & f(p_i) \\ (C, p_1,\dots,p_n) & \in & \overline{\mathcal{M}}_{g,n} & & \end{array}$$

(only (1)) :

$GW_{g,n,\beta}(\alpha_1,\dots,\alpha_n)$ $\qquad \alpha_i = P.D.(Z_i), \ Z_i \subset X$

$$= \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{virt}} \prod_{i=1} P_i^*(\alpha_i)$$

(virtual fund. class $\because$ dim $\overline{\mathcal{M}} \gneq$ exp. dim)

$$= \int_{[\overline{\mathcal{M}}_{g,n}]} \underbrace{F_*(\text{———}^{\shortparallel}\text{———})}_{I_{g,n,\beta}(\alpha_1,\dots,\alpha_n) \in H^*(\overline{\mathcal{M}}_{g,n})}$$

for imposing (2).

- For deforming smooth curve $C \subset X$

$$T^{Zar}_{[C]}\{C \subset X\} = \underset{\{\text{infinitesimal deformat}^{n}\}}{\overset{}{H^0(C, N_{C/X})}} \xrightarrow{k} \underset{\text{obstruction space.}}{H^1(C, N_{C/X})}$$

Moduli $\underset{}{\overset{\text{loc.}}{\sim}}$ $k^{-1}(0)$

expected dim $=$ dim $H^0(C, N_{C/X})$ $-$ dim $H^1(C, N_{C/X})$

$= \chi(C, N_{C/X})$ $\quad$ ($\because$ dim $C = 1$)

(Note: $0 \to T_C \to T_X|_C \to N_{C/X} \to 0$
$T_X^* = \Omega_X^1$ behaves better than T_X when C singular.

$\overline{\mathcal{M}}_{g,n}(X,\beta) \sim \mathrm{Ext}^1_C(f^*\Omega^1_X \to \Omega^1_C(\sum^n p_i), \ \mathcal{O}_C)$
$\xrightarrow{\kappa} \mathrm{Ext}^2_C(\ \text{———} \ \text{"} \text{————} \ , \ \text{—"—} \ )$

"glue" these, ... $\rightsquigarrow$ virtual fund. class
$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{virt} \in H_{\text{exp. dim.}}(\overline{\mathcal{M}}_{g,n}(X,\beta), \mathbb{Q})$

$\sim$ "Perfect obstruction theory"

deformat^n/obstruct^n $\underset{\text{by}}{\overset{\text{governed}}{\sim\!\sim}}$ H^* w/ only $\underset{\sim\sim}{2}$ nontrivial terms.

Other example: Deform stable bdl E/X^3 w/ $c_1(X) \geq 0$

$$H^1(X, \mathrm{End}_0 E) \xrightarrow{\kappa} H^2(X, \mathrm{End}_0 E)$$
inf. deformat^n obstruct^n

$H^0(X, \mathrm{End}_0 E) = 0$ ($\because$ E stable)
$H^{\geq 4}(\ \text{—"—} \ ) = 0$ ($\because$ $\dim X = 3$)
$H^3(\ \text{—"—} \ ) = H^0(X, \mathrm{End}_0 E \otimes K_X^{-1})^* = 0$ ($\because K_X^{-1} \geq 0$).

When X is CY3,
$H^2(X, \mathrm{End}_0 E) = H^1(X, \mathrm{End}_0 E)^* = T^*_{[E]}\mathcal{M}$

indeed $\kappa \in \mathrm{Sym}^2 T^*\mathcal{M}$

Geometric reason: $\mathcal{M} = \mathrm{Crit}(CS_C)$
(so expected dim of $\mathcal{M}$ is 0).
$\kappa \sim \nabla^2(CS_C)$ along $\mathcal{M}$.

"Symmetric obstruction theory"

Another example: Flat bdl. / oriented 3-mfd / $\mathbb{R}$.

Properties (axioms)

$$I_{g,n,\beta} : H^\bullet(X)^{\otimes n} \longrightarrow H^\bullet(\overline{\mathcal{M}}_{g,n})$$

- $I = 0$ unless β effective class
- $\deg I_{g,n,\beta}(\alpha_1,\dots) = 2(g-1)\dim X - 2\int_\beta c_1(X) + \sum_i |\alpha_i|$
- S_n equivariant
- $I_{g,n,\beta}(\alpha_1,\dots,\alpha_{n-1},\underbrace{[X]}_{\in H^0(X)}) = \pi_n^* I_{g,n-1,\beta}(\alpha_1,\dots,\alpha_{n-1})$ (i.e. no constraint on p_n)

  $\pi_n : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$ forget last point
- (Divisor) curve hitting divisor D is topological

  $$\pi_{n*} I_{g,n,\beta}(\alpha_1,\dots,\alpha_{n-1},[D]) = \left(\int_\beta D\right)\cdot I_{g,n-1,\beta}(\alpha_1,\dots,\alpha_{n-1})$$
- $I_{0,n,\underset{\text{classical}}{\underset{\Downarrow}{0}}}(-\alpha-) = \left(\int_X \prod_i \alpha_i\right)\cdot [\overline{\mathcal{M}}_{g,n}]$

  $g \geq 1 \rightsquigarrow$ const. maps are stable! (collapsing component ✓)
- (Splitting) 

  $$\varphi : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \longrightarrow \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$$

$$\varphi^* I_{g_1+g_2,n_1+n_2,\beta}(\alpha_1,\dots,\alpha_{n_1},\alpha_{n_1+1},\dots,\alpha_{n_1+n_2})$$

$$= \sum_{\beta=\beta_1+\beta_2}\sum_{i,j} g^{ij}\, I_{g_1,n_1+1,\beta_1}(\alpha_1,\dots,\alpha_{n_1},T_i) \otimes I_{g_2,n_2+1,\beta_2}(T_j,\alpha_{n_1+1},\dots,\alpha_{n_1+n_2})$$

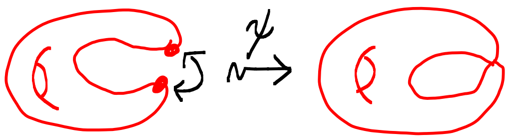
T_i : homog. basis of $H^\bullet(X)$

$(g_{ij}) = (\int_X T_i \wedge T_j)$, i.e. quadratic form $\overset{2}{\otimes} H^\bullet(X) \longrightarrow \mathbb{Q}$ as matrix wrt basis T_i's

$(g^{ij}) = (g_{ij})^{-1}$ namely $T^i := g^{ij} T_j$'s is dual base to T_i's.

- (Reduction)

  $\psi : \overline{\mathcal{M}}_{g-1,n+2} \longrightarrow \overline{\mathcal{M}}_{g,n}$



$$\psi^* I_{g,n,\beta}(-\alpha-) = \sum_{i,j} g^{ij}\, I_{g-1,n+2,\beta}(-\alpha-, T_i, T_j)$$

- I is invariant under deformations of complex str.
- (Motivic) I is induced by alg. cycle in $(X)^n \times \overline{\mathcal{M}}_{g,n}$

Eg. (Reconstruct^n) If $H^\bullet(X) = S^\bullet H^2(X)$

$$\langle I_{0,3,\beta} \rangle (\alpha_1, \alpha_2, \underset{H^2}{\underset{\Uparrow}{D}}) \Longrightarrow I_{0,\underset{\sim}{n},\beta}$$

$$\int_\beta c_1(X) \leq d+1$$

Eg. $\mathbb{P}^2$ GW for $g=0$

Enough to assume $\alpha_i = [pt]$ ($\because$ div. axiom)

$N_d \triangleq \langle I_{0,3d-1,d} \rangle ([pt]^{3d-1})$ # of genus 0 curves of deg d in $\mathbb{P}^2$ thru. $3d-1$ points.

Axioms & $\exists!$ line thru. 2 points

$\Longrightarrow$ recursive formula

$$N_d = \sum_{\substack{d=d_1+d_2 \\ d_i>0}} N_{d_1} \cdot N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right)$$

$$= 1,\ 1,\ 12,\ 620,\ 87304,\ \ldots\ldots$$

Eg. CY3 X^3

$$\text{v.}\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n}(X,\beta) = (1-g)\cancel{(\dim_{\mathbb{C}} X - 3)} + \cancel{\int_\beta c_1(X)} + n$$

$\Rightarrow$ no constraint. $\quad N_\beta := \langle I_{0,0,\beta} \rangle$

$$X = X^3(5) \subset \mathbb{CP}^4 \qquad \text{quintic CY3.}$$
$$= \{ s = 0 \} \qquad s \in H^0(\mathbb{CP}^4, \mathcal{O}(5))$$

$$\underset{\{s=0\}}{C \xrightarrow{f} \overset{}{X}} \iff \left\{ \begin{array}{l} C \xrightarrow{f} \mathbb{CP}^4 \curvearrowleft_s \mathcal{O}(5) \\ \underbrace{f^*(s)}_{\tilde{s}(f)} = 0 \in H^0(C, f^*\mathcal{O}(5)) \end{array} \right.$$

$$\begin{array}{ccc} & \mathcal{V}_d & \supset H^0(C, f^*\mathcal{O}_{\mathbb{P}^4}(5)) \simeq \mathbb{C}^{5d+1} \\ & \tilde{s} \uparrow\downarrow & \downarrow \\ \underbrace{\overline{\mathcal{M}}_{0,0}(X,d)}_{\{\tilde{s} = 0\}} \subset & \overline{\mathcal{M}}^{5d+1}_{0,0}(\mathbb{P}^4, d) & \ni f \end{array}$$

$$N_d = \int_{\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4,d)} c_{5d+1}(\mathcal{V}_d) \qquad \left(\begin{array}{c}\text{no } s \text{ anymore.} \\ \text{use localization} \\ T^4 \curvearrowright \mathbb{P}^4.\end{array}\right)$$

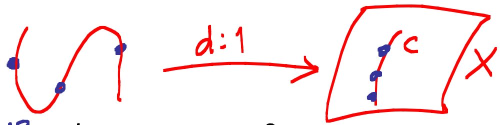
(issues on enumerative meaning)

Clemens conjecture: X generic quintic CY

- $\nexists$ family of rational curves
- disjoint
- $N_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$

(could be nodal !).

Multiple cover formula:



d^3 choices of preimage

$$\Rightarrow \quad N_d = \frac{1}{d^3} \sum_{k \mid d} \underbrace{n_k}_{\text{BPS \#}\ (\in \mathbb{Z}\,?)} k^3$$

Generating fu. $\displaystyle \sum_{d=1}^{\infty} N_d q^d = 5 + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d}$