Suggested Solution to Assignment 2

Exercise 2.1

1. By d'Alembert's formula, the solution is

$$\begin{aligned} u(x,t) &= \frac{1}{2}[e^{x+ct} + e^{x-ct}] + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin s ds \\ &= \frac{1}{2}[e^{x+ct} + e^{x-ct}] + \frac{1}{2c}[\cos(x-ct) - \cos(x+ct)]. \quad \square \end{aligned}$$

2. By d'Alembert's formula, the solution is

$$\begin{aligned} u(x,t) &= \frac{1}{2}\{\log[1+(x+ct)^2] + \log[1+(x-ct)^2]\} + \frac{1}{2c}\int_{x-ct}^{x+ct} (4+s)ds \\ &= \frac{1}{2}\{\log[1+(x+ct)^2] + \log[1+(x-ct)^2]\} + 4t + xt. \quad \square \end{aligned}$$

4. Define $v = u_t + cu_x$, then $v_t - cv_x = 0$. By the Geometric Method or Coordinate Method in Section 1.2, we obtain $v(x,t) = a(x+ct)$ and $u_t + cu_x = a(x+ct)$, which is a nonhomogeneous transport equation. Change variables $t' = x+ct$, $x' = x-ct$, then $u_{t'} = (ut + cu_x)/(2c) = a(t')/(2c)$. Thus $u = \int a(t')/(2c)dt' + b(x') = f(x+ct) + g(x-ct)$.

5. By d'Alembert's formula, the solution is

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = \frac{1}{2c}[\text{length of } (x-ct, x+ct) \cap (-a,a)].$$

So we have

$$u(x,a/2c) = \begin{cases} 0 & x \in (-\infty, -\frac{3a}{2}] \cup [\frac{3a}{2}, \infty); \\ \frac{1}{2c}(\frac{3a}{2} - x) & x \in [\frac{a}{2}, \frac{3a}{2}]; \\ \frac{a}{2c} & x \in [-\frac{a}{2}, \frac{a}{2}]; \\ \frac{1}{2c}(\frac{3a}{2} + x) & x \in [-\frac{3a}{2}, -\frac{a}{2}]; \end{cases} \quad u(x,a/c) = \begin{cases} 0 & x \in (-\infty, -2a] \cup [2a, \infty); \\ \frac{1}{2c}(2a - x) & x \in [0, 2a]; \\ \frac{1}{2c}(2a + x) & x \in [-2a, 0]; \end{cases}$$

$$u(x,3a/2c) = \begin{cases} 0 & x \in (-\infty, -\frac{5a}{2}] \cup [\frac{5a}{2}, \infty); \\ \frac{1}{2c}(\frac{5a}{2} - x) & x \in [\frac{a}{2}, \frac{5a}{2}]; \\ \frac{a}{c} & x \in [-\frac{a}{2}, \frac{a}{2}]; \\ \frac{1}{2c}(\frac{5a}{2} + x) & x \in [-\frac{5a}{2}, -\frac{a}{2}]; \end{cases} \quad u(x,2a/c) = \begin{cases} 0 & x \in (-\infty, -3a] \cup [3a, \infty); \\ \frac{1}{2c}(3a - x) & x \in [a, 3a]; \\ \frac{a}{c} & x \in [-a, a]; \\ \frac{1}{2c}(3a + x) & x \in [-3a, -a]; \end{cases}$$

$$u(x,5a/c) = \begin{cases} 0 & x \in (-\infty, -6a] \cup [6a, \infty); \\ \frac{1}{2c}(6a - x) & x \in [4a, 6a]; \\ \frac{a}{c} & x \in [-4a, 4a]; \\ \frac{1}{2c}(6a + x) & x \in [-6a, -4a]; \end{cases}$$

Here we omit the figures. $\square$

6.
$$\max_x u(x,t) = \begin{cases} t & 0 \le t \le \dfrac{a}{c}; \\ \dfrac{a}{c} & t \ge \dfrac{a}{c}. \end{cases} \quad \square$$

7. Since ϕ and ψ are odd function of x,

$$\begin{aligned} u(-x,t) &= \frac{1}{2}[\phi(-x+ct) + \phi(-x-ct)] + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(s)ds \\ &= \frac{1}{2}[-\phi(x-ct) - \phi(x+ct)] + \frac{1}{2c}\int_{x+ct}^{x-ct} \psi(-s)d(-s) \\ &= -\{\frac{1}{2}[\phi(x-ct) + \phi(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)d(s)\} = -u(x,t). \end{aligned}$$

Thus $u(x,t)$ is odd in x for all t. $\square$

8. (a) Change variables $v = ru$, then

$$v_{tt} = ru_{tt}, v_{rr} = (ru_r + u)_r = ru_{rr} + 2u_r,$$

which implies

$$v_{tt} = rc^2(u_{rr} + \frac{2}{r}u_r) = c^2 v_{rr}$$

(b) Using the same skill related to the wave equation(1), we have $v(r,t) = f(r+ct) + g(r-ct)$, where f and g are two arbitrary functions of a single variable. Hence $u = \frac{1}{r}f(r+ct) + \frac{1}{r}g(r-ct)$.

(c) Since $v(r,0) = r\phi(r)$ and $v_t(r,0) = r\psi(r)$ are both odd, we can extend v to all of $\mathbb{R}$ by odd reflection. That is, we set

$$\tilde{v}(r,t) = \begin{cases} v(r,t), & r > 0; \\ 0, & r = 0; \\ -v(-r,t), & r < 0. \end{cases}$$

Hence d'Alembert's formula implies

$$\tilde{v}(r,t) = \frac{1}{2}[(r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)] - \frac{1}{2c}\int_{r-ct}^{r+ct} s\psi(s)ds.$$

Therefore for $r > 0$,

$$u(r,t) = \frac{1}{r}v(r,t) = \frac{1}{2r}[(r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)] - \frac{1}{2cr}\int_{r-ct}^{r+ct} s\psi(s)ds. \quad \square$$

10. Using the same way above, since $(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t})(\frac{\partial}{\partial x} + 5\frac{\partial}{\partial t})u = 0$, we can obtain that the general solution is $u(x,t) = f(x + \frac{1}{4}t) + g(x - \frac{1}{5}t)$. The initial conditions implies

$$f(x) = \frac{1}{9}[4\phi(x) + 20\int_0^x \psi(s)ds + C],\ g(x) = \frac{1}{9}[5\phi(x) - 20\int_0^x \psi(s)ds - C].$$

Therefore, the solution is

$$u(x,t) = \frac{1}{9}[4\phi(x + \frac{1}{4}t) + 5\phi(x - \frac{1}{5}t)] + \frac{20}{9}\int_{x-\frac{1}{5}t}^{x+\frac{1}{4}t} \psi(s)ds. \quad \square$$

Exercise 2.2

1. By the law of conservation of energy, $E = \frac{1}{2}\int_{-\infty}^{\infty}(\rho u_t^2 + T u_x^2)\ dx$ is a constant independent of t. Since $\phi \equiv 0$ and $\psi \equiv 0$, we have $E \equiv 0$. Thus, the first vanishing theorem implies $u_t \equiv 0$ and $u_x \equiv 0$. So $u \equiv 0$ since $\phi \equiv 0$. $\square$

2. (a) By the chain rule,

$$\partial e/\partial t = u_t u_{tt} + u_x u_{xt},\ \partial e/\partial x = u_t u_{tx} + u_x u_{xx},$$
$$\partial p/\partial t = u_t u_{xt} + u_{tt} u_x,\ \partial p/\partial x = u_t u_{xx} + u_{tx} u_x.$$

Since $u_{tt} = u_{xx}$ and $u_{xt} = u_{tx}$,

$$\partial e/\partial t = \partial p/\partial x,\ \partial e/\partial x = \partial p/\partial t.$$

(b) From the result of (a),

$$e_{tt} = p_{xt} = p_{tx} = e_{xx},\ p_{tt} = e_{xt} = e_{tx} = p_{xx}.$$

So both $e(x,t)$ and $p(x,t)$ satisfy the wave equation. $\square$

3. (a) $(u(x-y,t))_{tt} = u_{tt}(x-y,t) = c^2 u_{xx}(x-y,t) = c^2\,(u(x-y,t))_{xx}$.
(b) $(u_x(x,t))_{tt} = u_{xtt}(x,t) = c^2 u_{xxx}(x,t) = c^2\,(u_x(x,t))_{xx}$.
(c) $(u(ax,at))_{tt} = a^2 u_{tt}(ax,at) = a^2 c^2 u_{xx}(ax,at) = c^2\,(u(ax,at))_{xx}$. $\square$

5. For damped string, $u_{tt} - c^2 u_{xx} + r u_t = 0$, where $c = \sqrt{\frac{T}{\rho}}$, the energy is

$$E = \frac{1}{2}\int_{-\infty}^{\infty} \rho(u_t^2 + c^2 u_x^2)dx.$$

Hence,

$$\begin{aligned} dE/dt &= \frac{1}{2}\int_{-\infty}^{\infty} \rho(2u_t u_{tt} + 2c^2 u_x u_{xt})dx \\ &= \int_{-\infty}^{\infty} \rho(c^2 u_t u_{xx} - r u_t^2 + c^2 u_x u_{xt})dx \\ &= \int_{-\infty}^{\infty} \rho(c^2 u_t u_{xx} - r u_t^2 - c^2 u_{xx} u_t)dx + (c^2 u_t u_x)\Big|_{-\infty}^{\infty} \\ &= -\int_{-\infty}^{\infty} \rho r u_t^2 dx \le 0. \quad \square \end{aligned}$$

6. (a)We compute that $u_{tt} = \alpha f''$, $u_r = \alpha' f - \alpha\beta' f'$, $u_{rr} = \alpha'' f - (2\alpha'\beta' + \alpha\beta'')f' + \alpha(\beta')^2 f''$. Plugging in the equations, we get

$$(\alpha'' + \frac{n-1}{r}\alpha')f - (2\alpha'\beta' + \alpha\beta'' + \frac{n-1}{r}\alpha\beta')f' + (\alpha(\beta')^2 - \frac{\alpha}{c^2})f'' = 0$$

(b)Setting the coefficients of f,f',f" to zero, we get $\alpha'' + \frac{n-1}{r}\alpha' = 0$, $2\alpha'\beta' + \alpha\beta'' + \frac{n-1}{r}\alpha\beta' = 0$, $\alpha(\beta')^2 - \frac{\alpha}{c^2} = 0$.
(c)From the third equation, we get $(\beta')^2 = 1/c^2$. Hence $\beta' = \pm 1/c$, $\beta'' = 0$. Then second equation simplifies to $2\alpha' + (n-1)\alpha/r = 0$. It has general solutions $\alpha(r) = Cr^{(1-n)/2}$, where C is a constant. Plugging back into the first equation, we get

$$\frac{1-n}{2}(\frac{1-n}{2} - 1) + (n-1)(\frac{1-n}{2}) = 0.$$

This implies $n = 1$ or $n = 3$. (d)If $n = 1$, $\alpha(r) = Cr^{(1-n)/2} \equiv Constant$.

Exercise 2.3

2. By the definition of maximum and minimum, $M(T)$ increases(i.e. nondecreasing) and $m(T)$ decreases(i.e. nonincreasing). $\square$

3. (a) Use the strong minimum principle, we omit the details here.

   (b) Use the minimum principle. Since $u(0,t) = u(1,t) = 0$, $u(x,t) \geq u(x,t_0)$ for $\forall t_0 \leq t < 1$. So $\mu(t)$ is dereasing.
   Or let the maximum occur at point $X(t)$, so that $\mu(t) = u(X(t),t)$. Differentiale $\mu(t)$, assuming that $X(t)$ is differentiable, we have
   $$\mu'(t) = u_x(X(t),t)X'(t) + u_t(X(t),t)$$
   Note at point $(X(t),t)$ we have $u_x = 0, u_{xx} \leq 0$. Hence, $\mu'(t) = u_{xx}(X(t),t) \leq 0$ and $\mu(t)$ is decreasing.

   (c) Here we omit the figure. Note that $u(0,t) = u(1,t) = 0$ and the result in (b). $\square$

4. (a) Note that $u(0,t) = u(1,t) = 0$ and $u(x,0) = 4x(1-x) \in [0,1]$. Then the conclusion can be verified by strong maximum principle.

   (b) Let $v(x,t) = u(1-x,t)$, then $v(0,t) = v(1,t) = 0$ and $v(x,0) = 4x(1-x) = u(x,0)$. Then the uniqueness theorem for the diffusion theorem implies $u(x,t) = u(1-x,t)$.

   (c)
   $$\frac{d}{dt}\int_0^1 u^2dx = \int_0^1 2uu_tdx = 2\int_0^1 uu_{xx}dx = -2\int_0^1 u_x^2dx.$$
   Since $u(x,t) > 0$ for all $t > 0$ and $0 < x < 1$, so u_x is not zero function. Hence, $\frac{d}{dt}\int_0^1 u^2dx < 0$ and $\int_0^1 u^2dx$ is a strictly decreasing function of t. $\square$

5. (a) We omit the details to verify that $u = -2xt - x^2$ is a solution. When t is fixed, u attains its maximum at $(-t,t)$ and $u(-t,t) = t^2$. So u attains its maximum at $(-1,1)$ in the closed rectangle $\{-2 \leq x \leq 2, 0 \leq t \leq 1\}$.

   (b) In our proof the maximum principle for the diffusion equation, the key point is that $v(x,t) = u(x,t) + \epsilon x^2$ satisfies $v_t - kv_{xx} < 0$. However, here $v_t - kv_{xx} = u_t - x(u+\epsilon x^2)_{xx} = -2\epsilon x$ so that the sign of $v_t - kv_{xx}$ is not unchanged in the closed rectangle $\{-2 \leq x \leq 2, 0 \leq t \leq 1\}$. $\square$

6. Let $w = u - v$ and use maximum principle for the diffusion equation. We omit the details. $\square$

7. (a) Let $w(x,t) = u(x,t) - v(x,t)$ and $w_\epsilon(x,t) = w(x,t) + \epsilon x^2$. Since $w_t - kw_{xx} = f - g \leq 0$, we can use the same method in the text book to derive the maximum principle for w. So $u \leq v$ at $x = 0$, $x = l$ and $t = 0$ implies $w \leq 0$ in the rectangle, i.e. $u \leq v$ for $0 \leq x \leq l$, $0 \leq t < \infty$. Here we omit the details of the method in the text book.

   (b) Let $u(x,t) = (1-e^{-t})\sin x$, and then $u_t - u_{xx} = \sin x$ and $u = 0$ at $x = 0$, $x = \pi$ and $t = 0$. Therefore, the result above implies $v(x,t) \geq (1-e^{-t})\sin x$. $\square$.

Extra 1. (1) Define $v(x,t) := e^{-at}u(x,t)$, then $v_t = kv_{xx}, V(0,t) = v(1,t) = 0, v(x,0) = sin(\pi x)$. By the Strong Maximum Principle, $0 < v(x,t) < 1, \forall t > 0, 0 < x < 1$. Thus, $0 < u(x,t) = e^{at}v(x,t) < 1, \forall t > 0, 0 < x < 1$

(2)Define $v(x,t) := u(1-x,t)$, then we can easily check that v solves the same problem as u. By the uniqueness of the solution, $u = v$

Extra 2. (a)Follow the proof of the Maximum Principle in the textbook. We only need to change the diffusion inequality (2) in Page 42 to be

$$v_t - kv_{xx} = u_t - ku_{xx} - 2\varepsilon k \leq -2\varepsilon k < 0$$

(b)Define $u(x,t) := v(x,t) - t\max_{-\infty<x<+\infty, 0<t<T} f(x,t)$, then

$$u_t - ku_{xx} = v_t - \max_{-\infty<x<+\infty,0<t<T} f(x,t) - kv_{xx} = f - \max_{-\infty<x<+\infty,0<t<T} f(x,t) \leq 0$$

$$\Rightarrow \max_{-\infty<x<+\infty,0\leq t\leq T} u(x,t) = \max_{-\infty<x<+\infty,t=0} u(x,t) = 0, by (a)$$

$$\Rightarrow v(x,t) \leq t \max_{-\infty<x<+\infty,0<t<T} f(x,t) \leq T \max_{-\infty<x<+\infty,0<t<T} f(x,t)$$

Exercise 2.4

1. By the general formula,

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-l}^{l} e^{-(x-y)^2/4kt} dy \\ &= \frac{1}{\sqrt{\pi}} \int_{(-l-x)/\sqrt{4kt}}^{(l-x)/\sqrt{4kt}} e^{-p^2} dp \\ &= \frac{1}{2}\{\mathscr{E}rf[\frac{x+l}{\sqrt{4kt}}] - \mathscr{E}rf[\frac{x-l}{\sqrt{4kt}}]\}. \quad \square \end{aligned}$$

2. By the general formula,

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-(x-y)^2/4kt} dy + \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{0} 3e^{-(x-y)^2/4kt} dy \\ &= \frac{1}{2} + \frac{1}{2}\mathscr{E}rf[\frac{x}{\sqrt{4kt}}] + \frac{3}{2} - \frac{3}{2}\mathscr{E}rf[\frac{x}{\sqrt{4kt}}] \\ &= 2 - \mathscr{E}rf[\frac{x}{\sqrt{4kt}}]. \quad \square \end{aligned}$$

3. By the solution formula (8)

$$\begin{aligned} u(x,t) =& \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/(4kt)} e^{3y} dy \\ =& \frac{e^{9kt+3x}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(y-6kt-x)^2/(4kt)} dy \\ =& e^{9kt+3x} \end{aligned}$$

Exercise 7 is used in the last equality.

5. Similar to Exercise 2.2.3.

8. By the definition of $S(x,t)$,

$$\max_{\delta\leq x<\infty} = \frac{1}{\sqrt{4\pi kt}} e^{-\delta^2/4kt},$$

so

$$\lim_{t\to 0^+} \max_{\delta\leq x<\infty} = \lim_{t\to 0^+} \frac{1}{\sqrt{4\pi kt}} e^{-\delta^2/4kt} = \lim_{x\to+\infty} \frac{\sqrt{x}}{\sqrt{4\pi k}} e^{-x\delta^2/4k} = 0. \quad \square$$

11. (a) Since $u(x,t)$ and $-u(-x,t)$ are the solutions and $u(x,0) = \phi(x) = -\phi(-x) = -u(-x,0)$, it follows from the uniqueness theorem that $u(x,t) = -u(-x,t)$.

    (b) Similar to (a).

    (c) Similar to (a). $\square$

14. Since
$$|e^{-(x-y)^2/4kt}\phi(y)| \le Ce^{-(x-y)^2/4kt+ay^2} = Ce^{(a-\frac{1}{4kt})y^2+\frac{x}{2kt}y-\frac{x^2}{4kt}},$$
$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\ dy$$
makes sense for $a - \frac{1}{4kt} < 0$, i.e. $0 < t < 1/(4ak)$, but not necessarily for large t, for example, $\phi(x) = e^{ax^2}$. $\square$

15. Suppose that both u and v are solution of the diffusion problem with the same Neumann boundary condition. Let $w(x,t) = u(x,t) - v(x,t)$, then w satisfies
$$w_t = kw_{xx}, \quad w(x,0) = w_x(0,t) = w_x(l,t) = 0.$$
Thus by the integration by part and the Neumann boundary condition,
$$\frac{d}{dt}\int_0^l \frac{1}{2}w^2(x,t)dx = -k\int_0^l w_x^2(x,t)dx \le 0.$$
Hence, the initial condition implies
$$\int_0^l \frac{1}{2}w^2(x,t)dx \le \int_0^l \frac{1}{2}w^2(x,0)dx = 0.$$
Therefor, $w = 0$, i.e. $u = v$ for all $t > 0$. $\square$

16. Let $v(x,t) = e^{bt}u(x,t)$, then v satisfies
$$v_t - kv_{xx} = 0, \quad v(x,0) = u(x,0) = \phi(x).$$
Hence, the general solution of v is
$$v(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\ dy,$$
and the general solution of u is
$$u(x,t) = \frac{e^{-bt}}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\ dy. \quad \square$$

18. Let $v(x,t) = u(x+Vt,t)$, then v satisfies
$$v_t - kv_{xx} = 0, \quad v(x,0) = u(x,0) = \phi(x).$$
Since
$$v(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\ dy,$$
$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-Vt-y)^2/4kt}\phi(y)\ dy. \quad \square$$

Exercise 2.5

1. Let $u(x,t) = -x^2 - (t-1)^2$ be the unique solution of the wave equation with boundary conditions:

$$u_{tt} = u_{xx}, \text{ for } -1 < x < 1, 0 < t < \infty,$$

$$u(x,0) = -x^2 - 1, \ u_t(x,0) = 2,$$

$$u(-1,t) = u(1,t) = -t^2 + 2t - 2.$$

But u attains its maximum 0 at $(0,1)$. $\square$