Tutorial 7 for MATH4220

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Attention: the midterm exam, which covers Chapter 1-3, will be scheduled on <u>13th Mar</u>(Tue).

1. Let $\phi(x)$ be a bounded piecewise continuous function, show that

$$\lim_{t\to 0^+} u(x,t) = \lim_{t\to 0^+}\int_{-\infty}^{+\infty} S(x-y,t)\phi(y)dy = \frac{1}{2}(\phi(x+)+\phi(x-)),$$

here $S(z,t)$ is the heat kernal.

Proof: Since

$$\frac{1}{\sqrt{4\pi}}\int_0^{\infty} e^{-p^2/4}dp = 1/2,$$

we have

$$\Big|\frac{1}{\sqrt{4\pi}}\int_0^{\infty} e^{-p^2/4}\phi(x+\sqrt{kt}p)dp - \frac{1}{2}\phi(x+)\Big| \le \frac{1}{\sqrt{4\pi}}\int_0^{\infty} e^{-p^2/4}|\phi(x+\sqrt{kt}p)-\phi(x+)|dp$$
$$\le \frac{1}{\sqrt{4\pi}}\int_{p_0}^{\infty} e^{-p^2/4}|\phi(x+\sqrt{kt}p)-\phi(x+)|dp + \frac{1}{\sqrt{4\pi}}\int_0^{p_0} e^{-p^2/4}|\phi(x+\sqrt{kt}p)-\phi(x+)|dp$$

For $\forall\epsilon>0$, choose p_0 large enough such that $\int_{p_0}^{\infty} e^{-p^2/4}dp$ is small enough and then

$$\frac{1}{\sqrt{4\pi}}\int_{p_0}^{\infty} e^{-p^2/4}|\phi(x+\sqrt{kt}p)-\phi(x+)|dp \le C\ max|\phi|\ \int_{p_0}^{\infty} e^{-p^2/4}dp < \frac{\epsilon}{2};$$

after this, we can choose t is small enough such that

$$|\phi(x+\sqrt{kt}p)-\phi(x+)| < \epsilon$$

and then

$$\frac{1}{\sqrt{4\pi}}\int_0^{p_0} e^{-p^2/4}|\phi(x+\sqrt{kt}p)-\phi(x+)|dp \le \left(\frac{1}{\sqrt{4\pi}}\int_0^{p_0} e^{-p^2/4}dp\right)\ \epsilon = \frac{\epsilon}{2}.$$

Hence,

$$\frac{1}{\sqrt{4\pi}}\int_0^{\infty} e^{-p^2/4}\phi(x+\sqrt{kt}p)\ dp \to \frac{1}{2}\phi(x+) \quad \text{as } t\searrow 0;$$

similarly we can prove that

$$\frac{1}{\sqrt{4\pi}}\int_{-\infty}^{0} e^{-p^2/4}\phi(x+\sqrt{kt}p)\ dp \to \frac{1}{2}\phi(x-) \quad \text{as } t\searrow 0.$$

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Finally, as $t \searrow 0$,

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{+\infty} S(x-y,t)\phi(y)dy \\ &= \int_{x}^{+\infty} S(x-y,t)\phi(y)dy + \int_{-\infty}^{x} S(x-y,t)\phi(y)dy \\ &= \frac{1}{\sqrt{4\pi}}\int_0^{\infty} e^{-p^2/4}\phi(x+\sqrt{kt}p)dp + \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{0} e^{-p^2/4}\phi(x+\sqrt{kt}p)dp \\ &\to \frac{1}{2}(\phi(x+)+\phi(x-)). \end{aligned}$$

2. Using the method of separation of variables to derive the solution formula for

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, 0 < x < l, t \in \mathbb{R} \\ u_x(0,t) = 0, u_x(l,t) = 0, \\ u(x,t=0) = \phi(x), \partial_t u(x,t=0) = \psi(x) \end{cases}$$

Solution: Step1: Let $u(x,t) = T(t)X(x)$, we have

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda,$$

which implies that λ is a constant. Moreover, the boundary conditions show that

$$X'(0) = X'(l) = 0.$$

Step2: Consider the eigenvalue problem

$$\begin{cases} X''(x) = -\lambda X(x), \\ X'(0) = X'(l) = 0. \end{cases}$$

First, claim that λ is nonnegative. In fact, multiply $X'' = -\lambda X$ by $\overline{X}$ and integrate from 0 to l, then

$$\int_0^l X''(x)\overline{X(x)}dx = -\lambda \int_0^l |X(x)|^2 dx.$$

Note that

$$\begin{aligned} \int_0^l X''(x)\overline{X(x)}dx &= X'(x)\overline{X(x)}\Big|_0^l - \int_0^l |X'(x)|^2 dx \\ &= -\int_0^l |X'(x)|^2 dx. \end{aligned}$$

Then

$$\lambda = \frac{\int_0^l |X'(x)|^2 dx}{\int_0^l |X(x)|^2 dx} \geq 0.$$

If $\lambda = 0$, then $X(x) =$constant.

If $\lambda = \beta^2 > 0$, then the general solution to $X'' = -\lambda X$ is

$$X(x) = a\cos(\beta x) + b\sin(\beta x)$$

with constants a, b. The boundary conditions give that

$$\begin{aligned} 0 &= X'(0) = b\beta && \Rightarrow b = 0, \\ 0 &= X'(l) = -a\beta\sin(\beta l) = 0 && \Rightarrow \beta = \frac{n\pi}{l}, n = 1, 2, \cdots \end{aligned}$$

Thus the eigenvalues and eigenfunctions are

$$\lambda_n = (\frac{n\pi}{l})^2, X_n(x) = \cos(\frac{n\pi}{l}x), n = 0, 1, \cdots$$

Step3: Solve the equation $T' = -\lambda c^2 T$, then

$$\begin{aligned} &\lambda_0 = 0, && T_0(t) = A_0 t + B_0 \\ &\lambda_n = (\frac{n\pi}{l})^2 > 0, && T_n(t) = A_n\sin(\frac{cn\pi}{l}t) + B_n\cos(\frac{cn\pi}{l}t), n = 1, 2, \cdots \end{aligned}$$

where $A_n, B_n, n = 0, 1, \cdots$ are constants to be determined.

Step4: Finally, the solution is given by

$$u(x,t) = A_0 t + B_0 + \sum_{n=1}^{\infty} A_n \sin(\frac{cn\pi}{l}t)\cos(\frac{n\pi}{l}x) + B_n\cos(\frac{cn\pi}{l}t)\cos(\frac{n\pi}{l}x).$$

While the initial data yield that

$$\begin{aligned} \phi(x) &= u(x,0) = B_0 + \sum_{n=1}^{\infty} B_n \cos(\frac{n\pi}{l}x), \\ \psi(x) &= \partial_t u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \frac{cn\pi}{l}\cos(\frac{n\pi}{l}x). \end{aligned}$$