Tutorial 4 for MATH4220

Rong ZHANG*

February 1, 2018

1. **Domain of dependence/ Domain of influence**

   Domain of dependence The value at point (x, t) depends on the value of ϕ at points $x - ct$, $x + ct$ and the value of ψ on interval $[x - ct, x + ct]$. See the Figure 2 on Page 38.

   Domain of influence An initial data at point $(x_0, 0)$ can affect the solution in region $\{(x, t) : x_0 - ct \leq x \leq x_0 + ct, t > 0\}$. See the Figure 1 on Page 38.

   The key is that waves propogate at finite speed.

2. **Application of Maximum Principle: Stability/Continuous dependence**

   Thoerem: Suppose $u_i(x, t), i = 1, 2$ are solutions of the following Initial-Boundary-Value-Problem:

$$\begin{cases} \partial_t u = k\partial_x^2 u \quad 0 \leq x \leq l, 0 \leq t \leq T \\ u(x, t = 0) = \phi_i(x) \quad 0 \leq x \leq l \\ u(x = 0, t) = g_i(t),\ u(x = l, t) = h_i(t) \quad 0 \leq t \leq T \end{cases}$$

   Then

$$\max_R |u_1(x, t) - u_2(x, t)| \leq \max\{\max_{0\leq x\leq l} |\phi_1(x) - \phi_2(x)|, \max_{0\leq t\leq T} |g_1(t) - g_2(t)|, \max_{0\leq t\leq T} |h_1(t) - h_2(t)|\}$$

   Proof: Set $v(x, t) = u_1(x, t) - u_2(x, t)$, the v satisfys

$$\partial_t v(x, t) = k\partial_x^2 v(x, t)$$

$$v(x, t = 0) = \phi_1(x) - \phi_2(x)$$

$$v(x = 0, t) = g_1(t) - g_2(t),\ v(x = l, t) = h_1(t) - h_2(t)$$

   Apply Maximum Principle to v, we have for any $0 < x < l, 0 < t < T$

$$\begin{aligned} v(x, t) \leq \max_{\partial_p R} v(x, t) &= \max\{\max_{0\leq x\leq l} \phi_1(x) - \phi_2(x), \max_{0\leq t\leq T} g_1(t) - g_2(t), \max_{0\leq t\leq T} h_1(t) - h_2(t)\} \\ &\leq \max\{\max_{0\leq x\leq l} |\phi_1(x) - \phi_2(x)|, \max_{0\leq t\leq T} |g_1(t) - g_2(t)|, \max_{0\leq t\leq T} |h_1(t) - h_2(t)|\} \end{aligned}$$

   Apply Minimum Principle to v, we have for any $0 < x < l, 0 < t < T$

$$v(x, t) \geq \min_{\partial_p R} v(x, t) = \min\{\min_{0\leq x\leq l} \phi_1(x) - \phi_2(x), \min_{0\leq t\leq T} g_1(t) - g_2(t), \min_{0\leq t\leq T} h_1(t) - h_2(t)\}$$

*Any questions on notes, please contact me at rzhang@math.cuhk.edu.hk

Then

$$-v(x,t) \leq \max\{\max_{0\leq x\leq l} \phi_2(x) - \phi_1(x), \max_{0\leq t\leq T} g_2(t) - g_1(t), \max_{0\leq t\leq T} h_2(t) - h_1(t)\}$$
$$\leq \max\{\max_{0\leq x\leq l} |\phi_1(x) - \phi_2(x)|, \max_{0\leq t\leq T} |g_1(t) - g_2(t)|, \max_{0\leq t\leq T} |h_1(t) - h_2(t)|\}$$

Hence

$$\max_R |v(x,t)| \leq \max\{\max_{0\leq x\leq l} |\phi_1(x) - \phi_2(x)|, \max_{0\leq t\leq T} |g_1(t) - g_2(t)|, \max_{0\leq t\leq T} |h_1(t) - h_2(t)|\}.$$

3. **The Plucked String**

For a vibrating string the speed is $c = \sqrt{\frac{T}{\rho}}$, consider an infinitely long string which satisfies the wave eqution

$$\partial_t^2 u - \partial_x^2 u = 0 \qquad -\infty < x < \infty$$

with the initial position

$$\phi(x) = \begin{cases} b - \dfrac{b|x|}{a} & |x| < a \\ 0 & |x| > a \end{cases}$$

and the initial velocity $\psi(x) = 0$ for all x. Sketch the string profile at each of the successive instants $t = \frac{a}{2c}, \frac{a}{c}, \frac{2a}{c}$.

Solution: The solution of this Cauchy problem is given by d'Alembert Formula

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)].$$

So we have

$$u(x, a/2c) = \begin{cases} 0 & x \in (-\infty, -\frac{3a}{2}] \cup [\frac{3a}{2}, \infty); \\ \frac{1}{2}(\frac{3b}{2} - \frac{b}{a}x) & x \in [\frac{a}{2}, \frac{3a}{2}]; \\ \frac{b}{2} & x \in [-\frac{a}{2}, \frac{a}{2}]; \\ \frac{1}{2}(\frac{3a}{2} + \frac{b}{a}x) & x \in [-\frac{3a}{2}, -\frac{a}{2}]; \end{cases}$$

$$u(x, a/c) = \begin{cases} 0 & x \in (-\infty, -2a] \cup [2a, \infty); \\ \frac{1}{2}(2b - \frac{b}{a}x) & x \in [a, 2a]; \\ \frac{b}{2a}|x| & x \in [-a, a]; \\ \frac{1}{2}(2b + \frac{b}{a}x) & x \in [-2a, 0]; \end{cases}$$

$$u(x, 2a/c) = \begin{cases} 0 & x \in (-\infty, -3a] \cup [-a, a] \cup [3a, \infty); \\ \frac{1}{2}(3b - \frac{b}{a}x) & x \in [2a, 3a]; \\ \frac{1}{2}(\frac{b}{a}x - b) & x \in [a, 2a]; \\ \frac{1}{2}(-b - \frac{b}{a}x) & x \in [-2a, -a]; \\ \frac{1}{2}(3b + \frac{b}{a}x) & x \in [-3a, -2a]; \end{cases}$$

Here we omit the figures. See the Figure 2 on Page 36.

The effect of the initial position $\phi(x)$ is a pair of waves travelling in either direction, one to the left $\frac{1}{2}\phi(x+ct)$ and another to the right $\frac{1}{2}\phi(x-ct)$, at the speed c and at half the original amplitude $\frac{b}{2}$. You can see this phenomenen by the graphs on page 36 clearly.