Tutorial 3 for MATH4220

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1. **Well-posedness** of the following linear algebraic equation

$$Ax = b$$

where A is a $m \times n$ matrix and $b \in \mathbb{R}^m$ is a vector. The data of this problem comprise the vector b.

(a) $m > n$: the number of equations is bigger than the number of unknowns, or, there are more rows than columns; this system is overdetermined. For given b, there is no solution.

(b) $m < n$: the number of equations is less than the number of unknowns, or, there are less rows than columns; this system is underdetermined. For given b, the solutions exist but are not unique.

(c) $m = n$:

- A is singular, or, $\det A = 0$, or, rank $A < m$.
  The solution is not unique, in fact, you may not find the solution in general.
- A is nonsingular, or, $\det A \neq 0$, or, rank $A = m$.
  There exists unique solution, but the system may be still ill-posed since it's unstable. Consider a matrix A with a small eigenvalue. If b is perturbed slightly, x will change greatly.

2. **Exercise**: Consider the Neumann problem

$$\Delta u = f(x, y, z) \quad \text{in } D$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on bdy } D$$

(a) What can we surely add to any solution to get another solution? So we don't have uniqueness.

(b) Use the divergence theorem and the PDE to show that

$$\iiint\limits_D f(x, y, z) dx\ dy\ dz = 0$$

is a necessary condition for the Neumann problem to have a solution.

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(c) Can you give a physical interpretation of part (a) and/or (b) for either heat flow or diffusion?

Solution:

(a) Adding a constant C to a solution will give another solution, so we do not have uniqueness if there is a solution;

(b) Integrating $f(x,y,z)$ on D and using the divergence theorem, we obtain

$$\iiint\limits_D f(x,y,z)dxdydz = \iiint\limits_D \Delta u dxdydz = \iiint\limits_D \nabla\cdot\nabla u dxdydz = \iint\limits_{\partial D} \nabla u\cdot n dS = 0$$

(c) For the heat flow, the equation which is independent of time t shows that the temperature u of the object reaches a steady state when there is an heat source or sink $f(x,y,z)$. At the same time, the Nuemann boundary condition means that the object is insulated,thus there is no heat flows in or out across the boundary. The part (b) shows that in order to make the PDE and the boundary condition hold simultaneously, we need $\iiint\limits_D f(x,y,z)dxdydz = 0$, that is, the total heat source or sink on the domain D should be 0 since the heat is steady and no heat flows in and out across the boundary.(otherwise it won't be steady if $\iiint\limits_D f(x,y,z)dxdydz \neq 0$). The part (a) means that if a given heat distribution is a steady state then rising or lowering the heat uniformly at every point is also a possible steady heat distribution. The difference between the steady state u and $u+C$ is that they have the different total heat energy.

3. **Classification** of the 2nd order linear PDE with constant coefficients:

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1 u_x + a_2 u_y + a_0 u = 0.$$

Show that by a linear transformation of the independent variables, the equation can be reduced to the following form

$$u_{xx} - u_{yy} + \cdots = 0$$

if $a_{12}^2 > a_{11}a_{22}$.

(a) $a_{11} \neq 0$ (or $a_{22} \neq 0$). Let $b_{12} = \frac{a_{12}}{a_{11}}$, $b_{22} = \frac{a_{22}}{a_{11}}$, then the equation reduces to

$$(\partial_x + b_{12}\partial y)^2 u + (b_{22} - b_{12}^2)u_{yy} + \cdots = 0.$$

Since $a_{12}^2 > a_{11}a_{22}$, $b_{22} - b_{12}^2 = \frac{a_{11}a_{22} - a_{12}^2}{a_{11}^2} < 0$, then let $b > 0$ such that $-b^2 = b_{22} - b_{12}^2$. Introduce the new variables by

$$x = \xi, \quad y = b_{12}\xi + b\eta$$

then $\partial_\xi = \partial_x + b_{12}\partial_y$, $\partial_\eta = b\partial_y$, and the equation becomes

$$u_{\xi\xi} - u_{\eta\eta} + \cdots = 0.$$

(b) $a_{11} = a_{22} = 0$, $a_{12} \neq 0$. Introduce the new variables by

$$\xi = x + y, \quad \eta = x - y$$

then $\partial_x = \partial_\xi + \partial_\eta, \partial_y = \partial_\xi - \partial_\eta$, and the equation becomes

$$u_{\xi\xi} - u_{\eta\eta} + \cdots = 0.$$