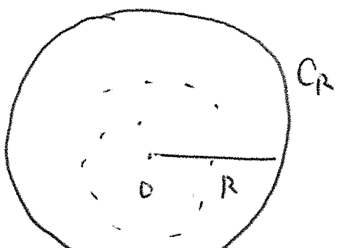
Lectures 18 and 19.

Cauchy Residue Thm helps us to calculate the integral $\int_C f(z)\,dz$. by checking its residues at singularities. Residue can be obtained by doing Laurent series expansion near each singularity. But if too many singularities are contained in the domain enclosed by curve C. calculation will be tedious, Here is another way to do it

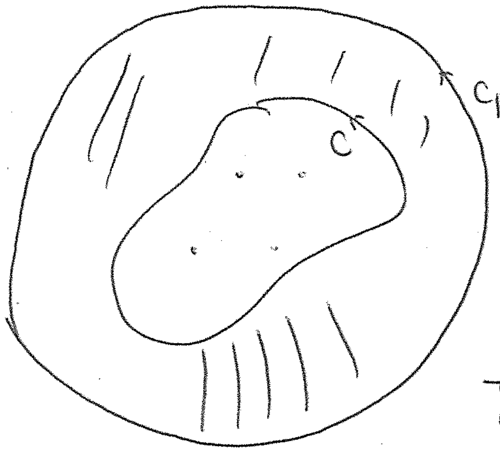
A1: 

all singularities are assumed to be included in B_R.

A2: $f(z)$ is analytic in $\mathbb{C}$. ~~ex~~ except these singularities

Then
$$\int_{C_R} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right)$$

Rk: for any curve C. s.t. all singularities of f are enclosed in C. then we can always find a $R>0$ large enough s.t the following relationship holds

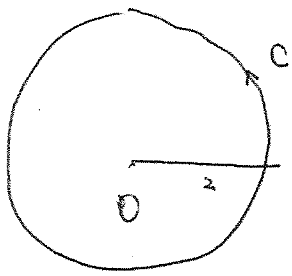


Since f is analytic in shaded Region. So $\int_C f(z)\,dz = \int_{C_R} f(z)\,dz$.

The above equality for integral on C_R can be reduced to integral on C.

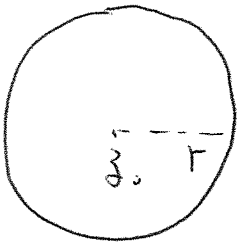
i.e. $\int_C f(z)\,dz = \int_{C_R} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} f(\frac{1}{z})$.

<u>ex:</u>



$$\int_C \frac{4z-5}{z(z-1)}\,dz.$$

The remainings are devoted to studing Classification of isolated singularities by Laurent series



f is analytic in $\{0 < |z - z_0| < r\}$

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n + \underbrace{\sum_{n=-\infty}^{-1} a_n (z - z_0)^n}_{\text{principle part.}}$$

Type I: no principal part. (removable singularity)

Type II. finite sum in principal part (poles)

Type III. Series sum in principal part (essential singularity)

for type I. $f(z) = a_0 + a_1 (z - z_0) + \cdots$

type I.1. $a_0 \neq 0$. z_0 is not a zero of f

type I.2. $a_0 = \cdots = a_m = 0$. $a_{m+1} \neq 0$.

$$f(z) = (z - z_0)^{m+1} \{ a_{m+1} + a_{m+2} (z - z_0) + \cdots \}$$

z_0 is $(m+1)$-th zero of f.

type I.3. $a_0 = \cdots = a_m = \cdots = 0$.

$f \equiv 0$.

From above arguments. a important property of zero of f is that all zeros of f must be isolated otherwise $f \equiv 0$ in Ω where Ω is a simply connected domain.

As for pole.
$$f(z) = \sum_{n=0}^{+\infty} a_n (z-z_0)^n + a_{-1} \cdot \frac{1}{z-z_0} + \cdots + a_{-m} \frac{1}{(z-z_0)^m}$$
$$= \frac{1}{(z-z_0)^m} \{ \underbrace{a_{-m} + a_{-(m-1)} (z-z_0) + \cdots}_{\phi(z)} \}$$

$\therefore$ $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$ with $\phi(z_0) \neq 0$.

$\therefore$ z_0 is called m-th order pole of f.

<u>Thm:</u>

z_0 is m-th order pole of f

$\Leftrightarrow$ $\exists$ ϕ analytic s.t. $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$.

Such representation helps us to calculate residue of f at m-th order pole z_0,

prop:
$$\operatorname*{Res}_{z=z_0} f = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

Finally we study behaviors of f near these singularities

Type I: $\lim_{z \to z_0} f(z) = a_0$ exists

if $f = a_0 + a_1(z - z_0) + \cdots$

Type II. $\lim_{z \to z_0} |f(z)| = \infty$. divergent

Type III. Weierstrauss Thm.

i.e. $\forall\, w_0 \in \mathbb{C}$ $\exists\, z_n \to z_0$ s.t

$$|f(z_n) - w_0| \to 0 \quad \text{as} \quad n \to +\infty.$$

provided that z_0 is an essential singularity of f.