lecture 1:

(I). Algebraic operations on complex numbers.

- summation: $(a_1+ib_1)+(a_2+ib_2)$

$$= (a_1+a_2)+i(b_1+b_2).$$

- commutative law:

$$z_1+z_2=z_2+z_1$$

- associative law:

$$z_1+(z_2+z_3)=(z_1+z_2)+z_3.$$

- "0"

$$z+(0+0i)=z. \Rightarrow 0=0+0i.$$

- summation inverse

$$z=a+ib. \Rightarrow -z=-a-ib.$$

- subtraction.

$$z_1-z_2=z_1+(-z_2).$$

- product.

$$(a+ib)(c+id)=(ac-bd)+i(bc+ad).$$

- "1"

$$(a+ib)(1+0i)=a+ib.$$

- product inverse

$$z=a+ib \Rightarrow z^{-1}=\frac{1}{z}=\frac{a}{a^2+b^2}-i\frac{b}{a^2+b^2}.$$

- Commutative law

$$z_1 z_2 = z_2 z_1$$

- associative law

$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

- Distributive law

$$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

- product $\Rightarrow$ Division

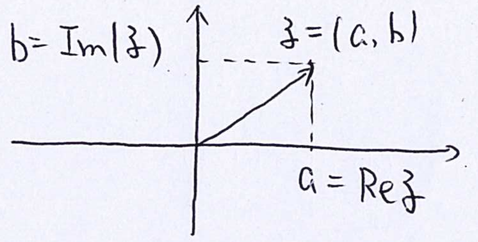
$$\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} = z_1 \cdot z_2^{-1}.$$

So. $$\frac{z_1 + z_2}{z_3} = \frac{z_1}{z_3} + \frac{z_2}{z_3}.$$

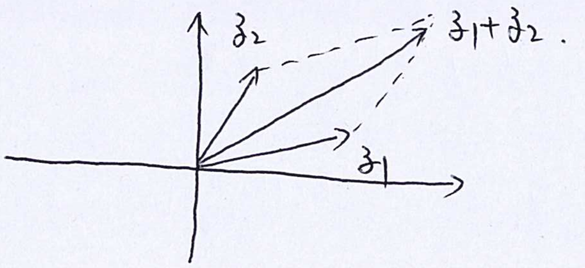
(II). Geometric Representation.

$$z = a + ib$$

$$\updownarrow$$

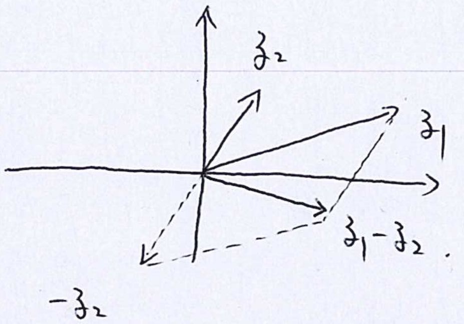


$$\Rightarrow \begin{cases} |\mathrm{Re}\, z| \le |z| \\ |\mathrm{Im}\, z| \le |z| \end{cases}.$$

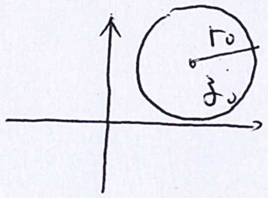


$\Rightarrow$ triangle inequality

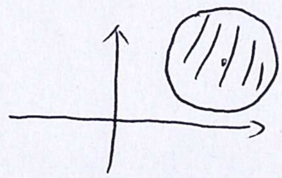
$$|z_1 + z_2| \le |z_1| + |z_2|$$



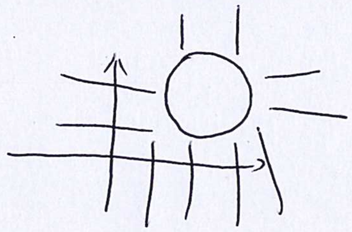
- Geometric Representation of Circle centring at z_0 with radius r_0



$\iff$ $|z - z_0| = r_0$

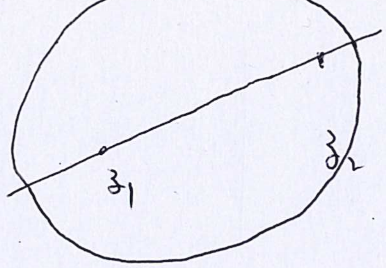


$\iff$ $|z - z_0| < r_0$ (interior)



$\iff$ $|z - z_0| > r_0$ (exterior)

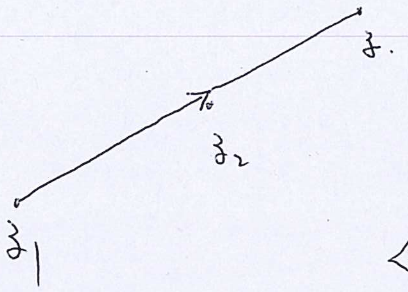
- ellipsis



$|z - z_1| + |z - z_2| = d.$

$d > 0$ is the length of long axis.

- Line. determined by z_1 and z_2.



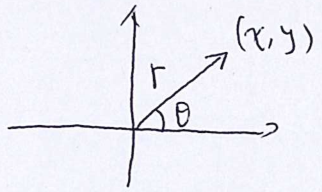
$\therefore \quad z - z_1 \parallel z_2 - z_1$

$\Longleftrightarrow \quad \exists\, k \quad \text{s.t} \quad z - z_1 = k(z_2 - z_1)$

k is real

$\Longleftrightarrow \quad \operatorname{Im}\left(\dfrac{z - z_1}{z_2 - z_1}\right) = 0.$

(III). exponential representation.



$$\therefore \quad x + iy = r\cos\theta + i\,r\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}.$$

Euler formula.

- $r = \sqrt{x^2 + y^2}$ is called modulus of $z = x + iy$.
- θ is called argument of $x + iy$.

Rk: r is uniquely decided. but θ is not.

They differ by $2k\pi$ with k an integer.

as a convention

$\operatorname{Arg}(z)$ is called principal argument with value restricted in $(-\pi, \pi]$.

$\arg(z) \triangleq \{ \operatorname{Arg}(z) + 2k\pi : k \text{ is an integer} \}$.

$\therefore$ $\arg(z_1) + \arg z_2 = \{ \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) + 2k\pi : k \text{ an integer} \}$.

Geometric Representation of product

$$z_1 = r_1 e^{i\theta_1} \qquad z_2 = r_2 e^{i\theta_2}$$

$$\therefore \quad z_1 z_2 = (r_1 r_2)\, e^{i(\theta_1 + \theta_2)}.$$