Math 2230A, Complex Variables with Applications

1. In each case, write the function f(z) in the form $f(z) = u(x, y) + iv(x, y)$: (a) $f(z) = z^3 + z + 1$; (b)$f(z) = \frac{\bar{z}^2}{z}(z \neq 0)$.
   *Suggestion:*In part (b), start by multiplying the numerator and dominator by $\bar{z}$.

2. Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use the expressions(see Sec.6)
   $$x = \frac{z + \bar{z}}{2} \quad and \quad y = \frac{z - \bar{z}}{2i}$$
   to write $f(z)$ in terms of z, and simplify the result.

3. Write the function
   $$f(z) = z + \frac{1}{z}$$
   in the form $f(z) = u(r, \theta) + iv(r, \theta)$.

4. Use definition (2), Sec.15, of limit to prove that
   $$(a) \lim_{z \to z_0} Rez = Rez_0; \quad (b) \lim_{z \to z_0} \bar{z} = \bar{z_0}; \quad (c) \lim_{z \to 0} \frac{\bar{z}^2}{z} = 0.$$

5. Show that the function
   $$f(z) = (\frac{z}{\bar{z}})^2$$
   has the value 1 at all nonzero points on the real and imaginary axes, where $z = (x, 0)$ and $z = (0, y)$, respectively, but that it has the value -1 at all nonzero points on the line $y = x$, where $z = (x, x)$. Thus show that the limit of $f(z)$ as z tends to 0 does not exist.[Note that it is not sufficient to simply consider nonzero points $z = (x, 0)$ and $z = (0, y)$, as it was in Example 2, Sec.15.]

6. Use definition (2), Sec.15, of limit to prove that
   $$\text{if } \lim_{z \to z_0} f(z) = \omega_0, \text{ then } \lim_{z \to z_0} |f(z)| = |\omega_0|.$$

7. With the aid of the theorem in Sec. 17, show that when
   $$T(z) = \frac{az + b}{cz + d} \quad (ad - bc \neq 0),$$
   (a) $\lim_{z \to \infty} T(z) = \infty$ if $c = 0$;

(b) $\lim_{z\to\infty} T(z) = \frac{a}{c}$ and $\lim_{z\to -d/c} T(z) = \infty$ if $c \neq 0$.

8. Use definition (3),Sec. 19, to give a direct proof that

$$\frac{d\omega}{dz} = 2z \text{ when } \omega = z^2.$$

9. Use the method in Example 2, Sec. 19, to show that $f'(z)$ does not exist at any point z when
(a) $f(z) = Rez$; (b)$f(z) = Imz$.

10. Let f denote the function whose values are

$$f(z) = \begin{cases} \overline{z}^2/z & \text{when} \quad z \neq 0 \\ 0 & \text{when} \quad z = 0 \end{cases}$$

Show that if z=0, then $\Delta\omega/\Delta z = 1$ at each nonzero point on the real and imaginary axes in the Δz, or Δx Δy, plane. Then show that $\Delta\omega/\Delta z = -1$ at each nonzero point $(\Delta x, \Delta x)$ on the line $\Delta y = \Delta x$ in the plane(Fig.29). Conclude from these observations that $f'(0)$ does not exist. Note that to obtain this result, it is not sufficient to consider only horizontal and vertical approaches to the origin in the Δz plane. (Compare with Exercise 5, Sec.18, as well as Example 2, Sec.19.)

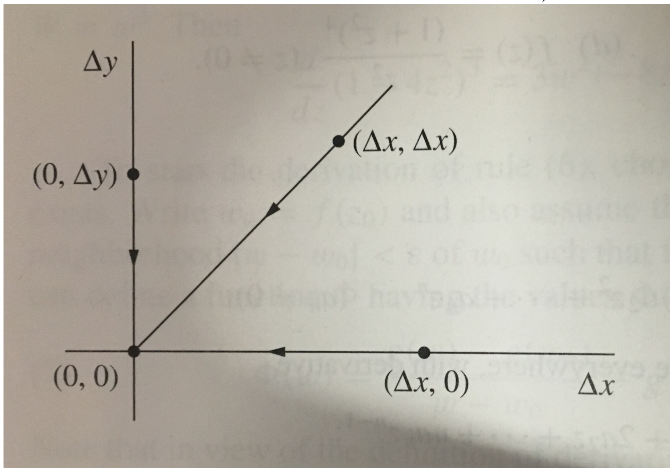

FIGURE 29