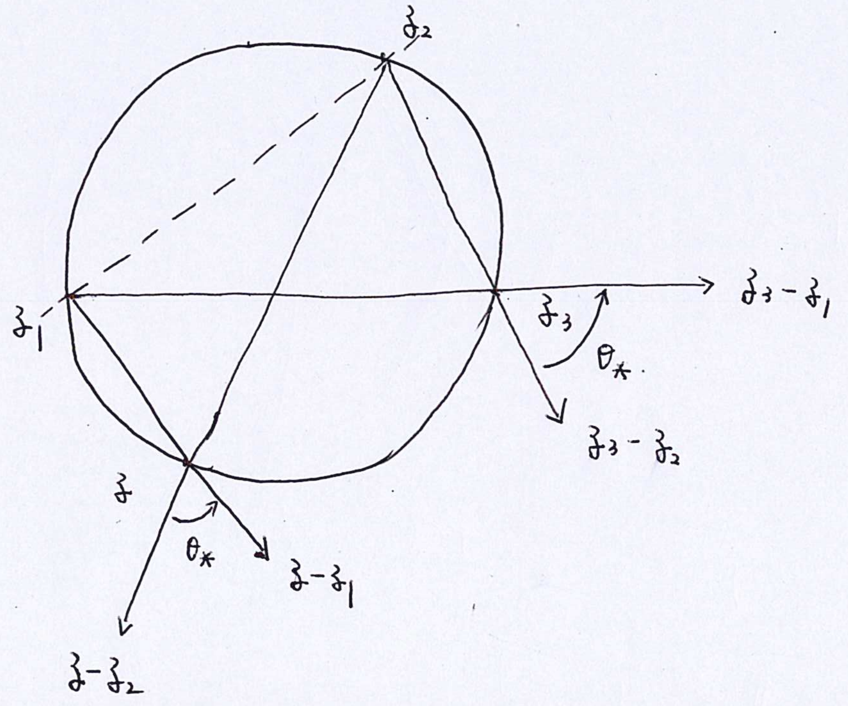
lecture 2:

· complex representation of a circle.

Case 1.



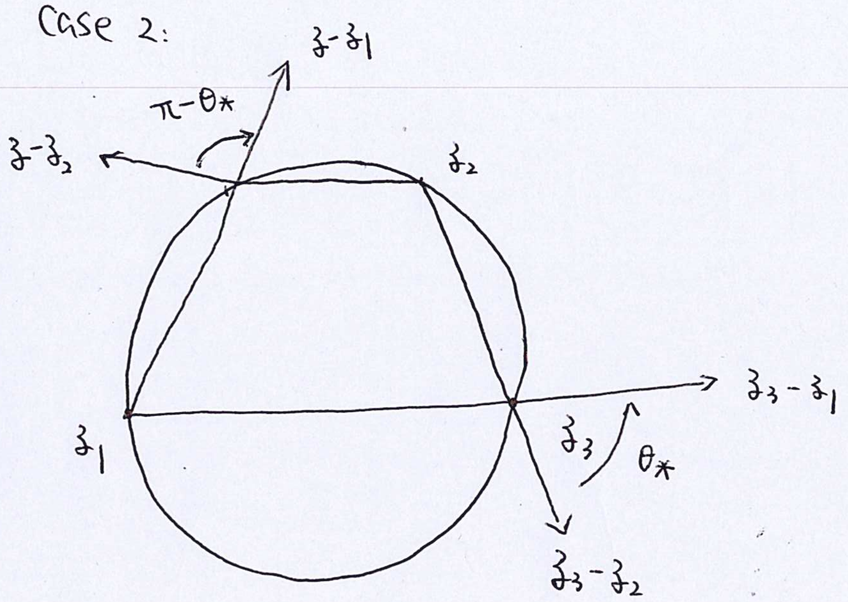

$$\Rightarrow \quad (z_3 - z_2) e^{i\theta_*} = r_1 (z_3 - z_1) \qquad r_1 > 0$$

$$(z - z_2) e^{i\theta_*} = r_2 (z - z_1) \qquad r_2 > 0.$$

$$\Rightarrow \quad \frac{z - z_1}{z - z_2} \Big/ \frac{z_3 - z_1}{z_3 - z_2} = \frac{r_1}{r_2} > 0.$$

$$\Rightarrow \quad \operatorname{Im}\left( \frac{z - z_1}{z - z_2} \Big/ \frac{z_3 - z_1}{z_3 - z_2} \right) = 0.$$

Case 2:




$\Rightarrow$

$$(z-z_2)\,e^{-(\pi-\theta_*)i} = r_2(z-z_1) \qquad r_2>0$$

$\Rightarrow$

$$r_2\,\frac{z-z_1}{z-z_2} = -e^{i\theta_*}$$

$\therefore$

$$\frac{z-z_1}{z-z_2}\bigg/\frac{z_3-z_1}{z_3-z_2} = -\frac{r_1}{r_2} < 0.$$

$\therefore$

$$\operatorname{Im}\left(\frac{z-z_1}{z-z_2}\bigg/\frac{z_3-z_1}{z_3-z_2}\right) = 0.$$

$\therefore$ Circle represented by $z_1, z_2, z_3$ is

$$\operatorname{Im}\left(\frac{z-z_1}{z-z_2}\bigg/\frac{z_3-z_1}{z_3-z_2}\right) = 0.$$

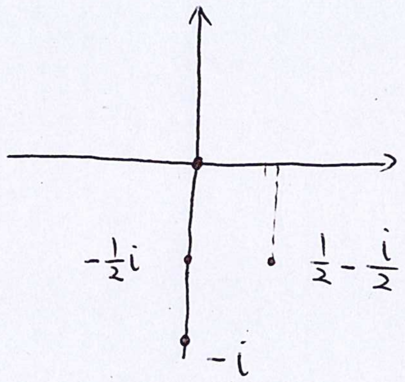
ex: $\quad \mathrm{Im}\left(\frac{1}{z}\right) = 1$

$$\mathrm{Im}\left(\frac{1}{z}\right) = 1 = \mathrm{Im}(i) \iff \mathrm{Im}\left(\frac{1}{z} - i\right) = \mathrm{Im}\left(\frac{1-iz}{z}\right) = 0.$$

$$\iff \mathrm{Im}\left(\frac{z+i}{z} \cdot (-i)\right) = 0.$$

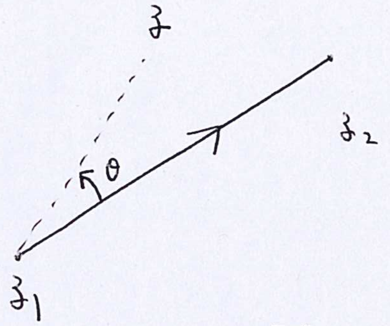
$\therefore \quad z_1 = -i \qquad z_2 = 0 \qquad \dfrac{z_3 - z_2}{z_3 - z_1} = -i.$

$\therefore \quad z_1 = -i, \qquad z_2 = 0. \qquad z_3 = \frac{1}{2} - \frac{i}{2}$




$$\left| z + \tfrac{1}{2} i \right| = \frac{1}{2} \qquad \text{Circle.}$$

• Side of Line




$\therefore \quad (z_2 - z_1)e^{i\theta} = r(z - z_1) \qquad r > 0.$

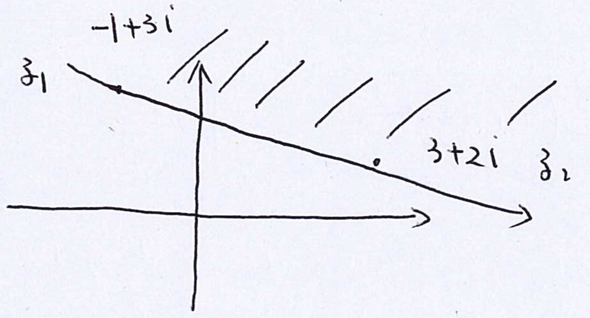
$\theta \in [0, \pi)$

$$\Rightarrow \quad \frac{z - z_1}{z_2 - z_1} = \frac{1}{r}(\cos\theta + i\sin\theta)$$

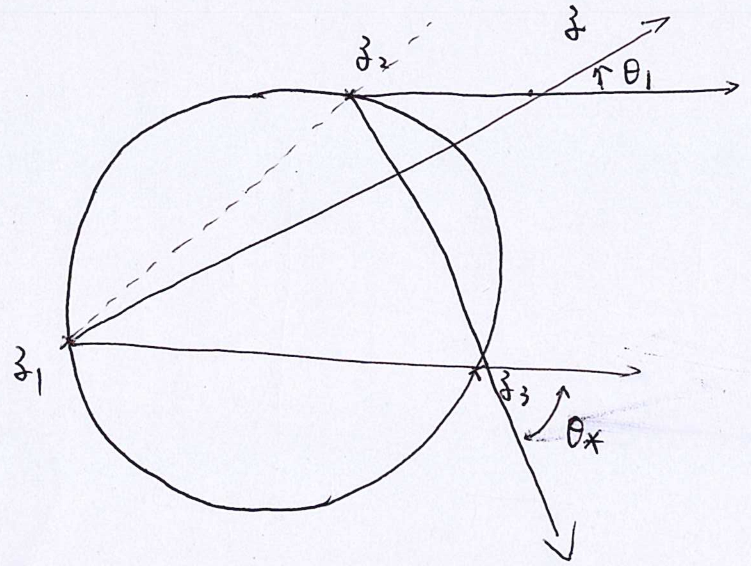
$$\therefore \quad \mathrm{Im}\left(\frac{z - z_1}{z_2 - z_1}\right) > 0.$$

this side is called left along $z_1 \to z_2$ direction.

ex: $\mathrm{Im}\left( \dfrac{z+1-3i}{4-i} \right) > 0.$




$z_1 = -1+3i \qquad z_2 = 3+2i$




① $z_1 \to z_2 \to z_2$ decide an orientation of circle

② suppose $z$ and $z_3$ on same side of line given by $z_1$ and $z_2$. and $z$ stays outside of the circle.

then $\theta_1 < \theta_*$.

$\therefore$

$$(z_3 - z_2)e^{i\theta_*} = r_1(z_3 - z_1) \qquad r_1 > 0.$$

$$(z - z_2)e^{i\theta_1} = r_2(z - z_1) \qquad r_2 > 0.$$

$\Rightarrow$

$$\frac{z-z_1}{z-z_2} \Big/ \frac{z_3-z_1}{z_3-z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_*)}$$

$\because\ \theta_1 < \theta_*. \quad \Rightarrow$

$$\mathrm{Im}\left(\frac{z-z_1}{z-z_2} \Big/ \frac{z_3-z_1}{z_3-z_2}\right) = -\frac{r_1}{r_2}\sin(\theta_*-\theta_1) < 0.$$

this is called right side of Circle.

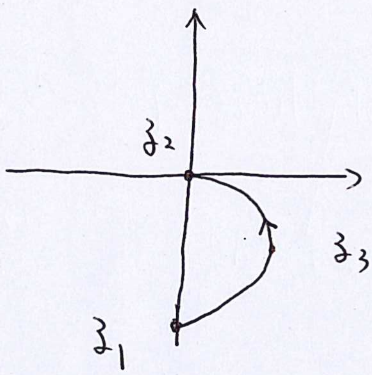
<u>Attn:</u> ① Howto find $z_1, z_2, z_3$

② Howto determine orientetion

③ $>0$ Left

$<0$ right

<u>ex:</u> $\mathrm{Im}\left(\frac{1}{z}\right) > 1$




$$\mathrm{Im}\left(\frac{1}{z}\right) > \mathrm{Im}(i)$$

$$\Leftrightarrow \quad \mathrm{Im}\left(\frac{1-iz}{z}\right) > 0.$$

$$\Leftrightarrow \quad \mathrm{Im}\left(\frac{z+i}{z}\cdot(-i)\right) > 0.$$

$$\Leftrightarrow \quad \mathrm{Im}\left(\frac{z+i}{z}\Big/ i\right) > 0.$$

$\therefore$ the ~~rig~~ region is inside the Circle.

other operations

① conjuge and ② root of a complex number.

ex: $(-16)^{1/4}$

③ power of a complex #.

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k}$$

$$(e^{i\theta})^n = e^{in\theta} \iff (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$