THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018 Fall MATH2230
Homework Set 10 (Due on Nov. 26)

All the homework problems are taken from Complex Variables and Applications, Ninth Edition, by James Ward Brown/Ruel V. Churchill.

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1 In each case, write the principal part of the function at its isolated singular point and determine whether that point is a removable singular point, an essential singular point, or a pole:

(a) $z\exp(1/z)$; (b) $\dfrac{z^2}{1+z}$; (c) $\dfrac{\sin z}{z}$; (d) $\dfrac{\cos z}{z}$; (e) $\dfrac{1}{(2-z)^3}$.

2 Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

(a) $\dfrac{1-\cosh z}{z^3}$; (b) $\dfrac{1-e^{2z}}{z^4}$; (c) $\dfrac{e^{2z}}{(z-1)^2}$.

3 Suppose that a function f is analytic at z_0 and write $g(z) = f(z)/(z-z_0)$. Show that
(a) if $f(z_0) \neq 0$, then z_0 is a simple pole of g, with residue $f(z_0)$;
(b) if $f(z_0) = 0$, then z_0 is a removable singular point of g.
Suggestion: As pointed out in Sec. 62, there is a Taylor series for $f(z)$ about z_0 since f is analytic there. Start each part of this exercise by writing out a few terms of that series.

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3 In each case, find the order m of the pole and the corresponding residue B at the singularity $z = 0$;

(a) $\dfrac{\sinh z}{z^4}$; (b) $\dfrac{1}{z(e^z-1)}$.

5 Find the value of the integral

$$\int_C \frac{dz}{z^3(z+4)}$$

taken counterclockwise around the circle (a) $|z| = 2$; (b) $|z+2| = 3$.

6 Evaluate the integral

$$\int_C \frac{\cosh \pi z dz}{z(z^2+1)}$$

when C is the circle $|z| = 2$, described in the positive sense.

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5 Let C denote the positively oriented circle $|z| = 2$ and evaluate the integral

(a) $\displaystyle\int_C \tan z dz$; (b) $\displaystyle\int_C \frac{dz}{\sinh 2z}$.

6 Let C_N denote the positively oriented boundary of the square whose edges lie along the lines

$$x = \pm\left(N + \frac{1}{2}\right)\pi \quad \text{and} \quad y = \pm\left(N + \frac{1}{2}\right)\pi,$$

where N is a positive integer. Show that

$$\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right]$$

Then, using the fact that the value of this integral tends to zero as N tends to infinity (Exercise 8. Sec. 47), point out how it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$