THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018 Fall MATH2230
Homework Set 6(Due on)

All the homework problems are taken from Complex Variables and Applications, Ninth Edition, by James Ward Brown/Ruel V. Churchill.

P132-135

For the functions f and contours C in Exercises 1 through 8, use parametric representations for C, or legs of C, to evaluate

$$\int_C f(z)dz,$$

1. $f(z) = \dfrac{z+2}{z}$ and C is
   (a) the semicircle $z = 2e^{i\theta}$ $(0 \le \theta \le \pi)$;
   (b) the semicircle $z = 2e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;
   (c) the circle $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$.
2. $f(z) = z - 1$ and C is the arc from $z = 0$ to $z = 2$ consisting of
   (a) the semicircle $z = 1 + e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;
   (b) the segment $z = x$ $(0 \le x \le 2)$ of the real axis.
3. $f(z) = \pi \exp(\pi \bar{z})$ and C is the boundary of the square with vertices at the points $0, 1, 1+i$ and i, the orientation of C being in the counterclockwise direction.
4. $f(z)$ is defined by means of the equations

$$f(z) = \begin{cases} 1 & \text{when } y < 0\ , \\ 4y & \text{when } y > 0. \end{cases}$$

   and C is the arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$.
5. $f(z) = 1$ and C is an arbitrary contour from any fixed point z_1 to any fixed point z_2 in the z plane.
6. $f(z)$ is the principal branch

$$z^i = \exp(iLogz) \ \ (|z| > 0, -\pi < \text{Arg}(z) < \pi)$$

   of the power function z^i and C is the semicircle $z = e^{i\theta}$ $(0 \le \theta \le \pi)$.
7. $f(z)$ is the principal branch

$$z^{-1-2i} = \exp[(-1-2i)Logz] \ \ (|z| > 0, -\pi < \text{Arg}(z) < \pi)$$

   of the indicated power function, and C is the contour

$$z = e^{i\theta} \ \ (0 \le \theta \le \pi/2)$$

8. $f(z)$ is the principal branch

$$z^{a-1} = \exp[(a-1)Logz] \;\; (|z| > 0, -\pi < \text{Arg}(z) < \pi)$$

of the power function z^{a-1}, where a is a nonzero real number, and C is the positively oriented circle of radius R about the origin.

9. Let C denote the positively oriented unit circle about the origin.

   (a) Show that if $f(z)$ is the principal branch

   $$z^{-3/4} = \exp\left[-\frac{3}{4}Logz\right] \;\; (|z| > 0, -\pi < \text{Arg}(z) < \pi)$$

   of $z^{-3/4}$, then

   $$\int_C f(z)dz = 4\sqrt{2}i.$$

   (b) Show that if $g(z)$ is the branch

   $$z^{-3/4} = \exp\left[-\frac{3}{4}Logz\right] \;\; (|z| > 0, 0 < \text{Arg}(z) < 2\pi)$$

   of the same power function as in part (a), then

   $$\int_C g(z)dz = -4 + 4i.$$

10. With the aid of the result in Exercise 3. Sec. 42. evaluate the integral

$$\int_C z^m \overline{z}^n dz,$$

where m and n are integers and C is the unit circle $|z| = 1$, taken counterclockwise.

11. Let C denote the semicircular path shown in Fig. 46. Evaluate the integral of the function $f(z) = \overline{z}$ along C using the parametric representation (see Exercise 2. Sec. 43)

   (a) $z = 2e^{i\theta} \;\; (-\pi/2 \leq \theta \leq \pi/2)$;

   (b) $z = \sqrt{4 - y^2} + iy \; (-2 \leq y \leq 2)$.

12. (a) Suppose that a function $f(z)$ is continuous on a smooth arc C, which has a parametric representation $z = z(t)$ $(a \leq t \leq b)$; that is, $f[z(t)]$ is continuous on the interval $a \leq t \leq b$. Show that if $\phi(\tau)$ $(\alpha \leq t \leq \beta)$ is the function described in Sec. 43. then

   $$\int_a^b f[z(t)]z'(t)dt = \int_\alpha^\beta f[Z(\tau)]Z'(\tau)d\tau$$

   where $Z(\tau) = z[\phi(\tau)]$.

(b) Point out how il follows that the identity obtained in part (a) remains valid when C is any contour, not necessarily a smooth one, and $f(z)$ is piecewise continuous on C. Thus show that the value of the integral of $f(z)$ along C is the same when the representation $z = Z(\tau)$ $(\alpha \le \tau \le \beta)$ is used, instead of the original one. Suggestion: In part (a), use the result in Exercise 1(b). Sec. 43. and then refer to expression (14) in that section.

13. Let C_0 denote the circle centered at z_0 with radius R, and use the parametrization

$$z = z_0 + Re^{i\theta} \ \ (-\pi \le \theta \le \pi)$$

to show that

$$\int_{C_0)} (z - z_0)^{n-1} dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2... \ , \\ 2\pi i & \text{when } n = 0. \end{cases}$$

(Put $z_0 = 0$ and then compare the result with the one in Exercise 8 when the constant a there is a nonzero integer.)

P138-139:

1. Without evaluating the integral, show that

   (a) $\left| \int_C \frac{z+4}{z^3-1} dz \right| \le \frac{6\pi}{7}$

   (b) $\left| \int_C \frac{dz}{z^2-1} dz \right| \le \frac{\pi}{3}$

   when C is the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$.

2. Let C denote the line segment from $z = i$ to $z = 1$ (fig. 49), and show that

$$\left| \int_C \frac{dz}{z^4} dz \right| \le 4\sqrt{2}$$

   without evaluating the integral.

3. Show that if C is the boundary of the triangle with vertices at the points $0, 3i$ and -4, oriented in the counterclockwise direction (see fig. 50), then

$$\left| \int_C (e^z - \overline{z}) dz \right| \le 60$$

4. Let C_R denote the upper half of the circle $|z| = R$ $(R > 2)$, taken in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4} dz \right| \le \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}$$

   Then, by dividing the numerator and denominator on the right here by R^4, show that the value of the integral tends to zero as R tends lo infinity. (Compare with Example 2 in Sec.47.)

5. Let C_R be the circle $|z| = R(R > 1)$, described in the counterclockwise direction, Show that

$$\left| \int_{C_R} \frac{Log z}{z^2} dz \right| \leq 2\pi \frac{\pi + \ln R}{R}$$

and then use L' Hospital's rule to show that the value of this integral tends to zero as R tends lo infinity.

6. Let C_ρ denote a circle $|z| = \rho$ $(0 < \rho < 1)$, oriented in the counterclockwise direction, and suppose that $f(z)$ is analytic in the disk $|z| \leq 1$. Show that if $z^{-1/2}$ represents any particular branch of that power of z, then there is a nonnegative constant M, independent of ρ, such that

$$\left| \int_{C_\rho} z^{-1/2} f(z) dz \right| \leq 2\pi M \sqrt{\rho}.$$

Thus show that the value of the integral here approaches 0 as ρ tends to 0.