MATH 2230 HW2

P30-31

1a $\quad 2i = 2e^{\frac{\pi}{2}i + 2k\pi i} \qquad k \in \mathbb{Z}$

By definition, $(2i)^{\frac{1}{2}} = e^{\frac{1}{2}\log(2i)}$

$$= e^{\frac{1}{2}\left(\log 2 + i\left(\frac{\pi}{2} + 2k\pi\right)\right)}$$

$$= \sqrt{2}\, e^{i\left(\frac{\pi}{4} + k\pi\right)}$$

$$= \sqrt{2}\, e^{\frac{\pi}{4}i},\ \sqrt{2}\, e^{i\frac{5\pi}{4}}$$

$$= \pm(1+i)$$

1b $\quad 1 - \sqrt{3}i = 2e^{-\frac{\pi}{3}i + 2k\pi i}$

$$(1-\sqrt{3}i)^{\frac{1}{2}} = e^{\frac{1}{2}\left(\log 2 + i\left(\frac{-\pi}{3} + 2k\pi\right)\right)}$$

$$= \sqrt{2}\, e^{i\left(\frac{-\pi}{6} + k\pi\right)}$$

$$= \pm \frac{1}{\sqrt{2}}\left(\sqrt{3} - i\right)$$

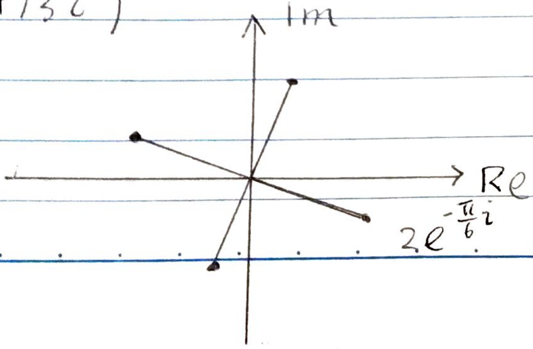
3 $\quad -8 - 8\sqrt{3}i = -16\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 16e^{-\frac{2\pi}{3}i + 2k\pi i}$

$$(-8-8\sqrt{3}i)^{\frac{1}{4}} = 2e^{-\frac{\pi}{6}i + \frac{k\pi i}{2}}$$

$$= 2e^{-\frac{\pi}{6}i},\ 2e^{-\frac{\pi}{6}i + \frac{\pi i}{2}},\ 2e^{-\frac{\pi}{6}i + \pi i},\ 2e^{-\frac{\pi}{6}i + \frac{3\pi}{2}}$$

$$= \pm(\sqrt{3} - i),\ \pm(1 + \sqrt{3}i)$$

$2e^{-\frac{\pi}{6}i} = \sqrt{3} - i$ is principal



P35

4c $\dfrac{1}{z} = \dfrac{1}{x+iy} = \dfrac{x-iy}{x^2+y^2}$

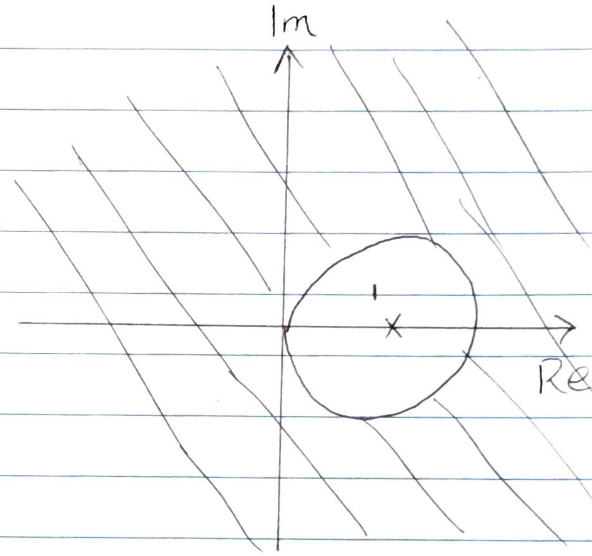
~~If Re(z) Re$\left(\frac{1}{z}\right) \le 1/2$, then~~

If $\text{Re}\left(\dfrac{1}{z}\right) \le 1/2$, then

$$\frac{x}{x^2+y^2} \le 1/2$$

$$0 \le x^2+y^2-2x$$

$$1 \le (x-1)^2+y^2$$



P89

1a $e^{2\pm 3\pi i} = e^2 e^{\pm 3\pi i}$

$= e^2 e^{\pm \pi i}$

$= -e^2$

1b $e^{\frac{2+\pi i}{4}} = e^{1/2} e^{\frac{\pi}{4} i}$

$= e^{1/2}\left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}} i\right)$

$= \sqrt{\dfrac{e}{2}}\,(1+i)$

1c $e^{z+\pi i} = e^z \cdot e^{\pi i}$

$= -e^z$.

16 $2 = \cos z = \cos x \cosh y - i \sin x \sinh y$

$\Rightarrow \begin{cases} 2 = \cos x \cosh y \\ 0 = \sin x \sinh y \end{cases}$

then $\sin x = 0$ or $\sinh y = 0$ (If $\sinh y = 0$, $y = 0$, then no solution for $\cos x = 2$)

If $\sin x = 0 \Rightarrow x = k\pi \quad k \in \mathbb{Z}$

then $2 = (-1)^k \cosh y$

Since $\cosh y = \dfrac{e^y + e^{-y}}{2}$, then

$(e^y)^2 - (-1)^k 4 e^y + 1 = 0$

But the equation has no real solution if k is odd

$\Rightarrow k = 2n,\ n \in \mathbb{Z}$ ~~and~~ $\cancel{e^y = \frac{4 \pm \sqrt{16-4}}{2}}$

$\Rightarrow y = \cosh^{-1} 2$

$\Rightarrow z = 2n\pi + i \cosh^{-1} 2.$

If $y = \cosh^{-1} 2 \Rightarrow (e^y)^2 - (4) e^y + 1 = 0$

$e^y = \dfrac{4 \pm \sqrt{16-4}}{2}$

$= 2 \pm \sqrt{3}$

$y = \log(2 \pm \sqrt{3})$

~~$2+\sqrt{3} = \sqrt{7}\, e^{\pm\theta i}$~~ ~~where the $\tan\theta = \frac{\sqrt{3}}{2}$ for $0 \le \theta < \frac{\pi}{2}$~~

$$2-\sqrt{3} = \frac{1}{2+\sqrt{3}}$$

$$\Rightarrow \quad z = 2n\pi + i\cosh^{-1} 2$$

$$= 2n\pi + i\log(2 \pm \sqrt{3})$$

$$= 2n\pi \pm i\log(2+\sqrt{3}).$$

P112

16a $\quad i = \sinh z = \sinh x \cos y + i\cosh x \sin y$

$$\begin{cases} \sinh x \cos y = 0 \\ \cosh x \sin y = 1 \end{cases}$$

If $\cos y = 0$, then $y = \pi/2 + \pi k \quad k \in \mathbb{Z}$.

$$\cosh x\,(-1)^k = 1$$

$$\Rightarrow \quad e^x + e^{-x} = 2(-1)^k$$

$$(e^x)^2 - 2(-1)^k e^x + 1 = 0$$

So k is even, $k = 2n$,

$$e^x = 1 \Rightarrow x = 0$$

$$\Rightarrow \quad z = (2n + 1/2)\pi i$$

If $\sinh x = 0$, $x = 0 \Rightarrow \sin y = 1 \Rightarrow y = (2n + 1/2)\pi$.

16b $\quad 1/2 = \cosh z = \cosh x \cos y + i \sinh x \sin y$

$$\Rightarrow \begin{cases} 1/2 = \cosh x \cos y \\ 0 = \sinh x \sin y \end{cases}$$

If $\sin y = 0 \Rightarrow y = \cancel{n\pi}\ k\pi,$

$$1/2 = \cosh x \,(-1)^k$$

$$(e^x)^2 - (-1)^k e^x + 1 = 0 \Rightarrow \cancel{k=2m} \text{ no real solution.}$$

If $\sinh x = 0 \Rightarrow x = 0$

$\Rightarrow \cos y = 1/2 \Rightarrow y = \pm \pi/3 + 2n\pi \Rightarrow z = 2n\pi \pm \pi/3$

17 $\quad \cosh z = -2$

$$e^z + e^{-z} = -4$$

$$(e^z)^2 + 4e^z + 1 = 0$$

$$e^z = \frac{-4 \pm \sqrt{16-4}}{2}$$

$$= -2 \pm \sqrt{3}$$

$$= -(2 \pm \sqrt{3})$$

$$= e^{\log(2\pm\sqrt{3})} \cdot e^{(\pi + 2n\pi)i}$$

$$\Rightarrow z = \log(2 \pm \sqrt{3}) + (2n+1)\pi i$$

$$= \pm \log(2+\sqrt{3}) + (2n+1)\pi i$$