Lecture 12

Network Flow Problems

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Agenda

1. **Maximal Flow Problem**
2. **Methods for Maximal-Flow Problems**
3. **Maximal Flow and Minimal Cut**
4. **LP Interpretation of Max-flow Min-cut Problem**

Maximal Flow Problem

Definition

Let Q be the set of all distinct *ordered* pairs of elements of a set V, that is,

$$Q = \{(x_i, x_j) \mid x_i \in V, x_j \in V\}$$

The pair $G = (V, E)$ with $E \subset Q$, is called a *directed graph*, the elements of E are called *directed edges*.

Definition

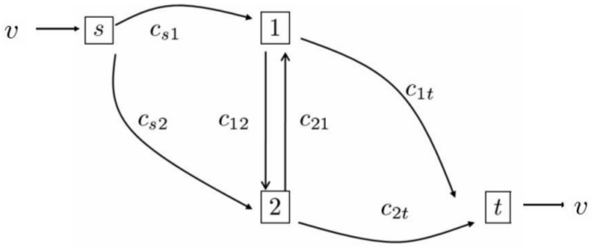
An incidence matrix can be defined for a directed graph. Let $A = (a_{ij})$, $i = 1, \cdots, |V|$, $j = 1, \cdots, |E|$ be the incidence matrix for a directed graph $G(V, E)$ defined as follows

$$a_{ij} = \begin{cases} -1 & \text{if } e_j = (x_k, x_i), k \neq i, \\ 1 & \text{if } e_j = (x_i, x_k), k \neq i, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

- The transshipment problem is a special class of network flow problems. To be more specific, we consider the problem of shipping a certain homogeneous commodity from a specified origin, called the *source*, to a particular destination, called the *sink*.
- The flow network will generally consist of some intermediate vertex, known as *transshipment points*, through which the flows are rerouted.
- At the transshipment points we impose the condition of *conservation of flow*, i.e. what is shipped into it is shipped out.

Example

Consider a flow network given by the following diagram. Vertex s is the source and vertex t is the sink. The number c_{ij} on edge (i, j) represents the capacity of that edge.



Let f_{ij} be the flow in edge (i, j) and f be the total flow from the source s to the sink t. The maximal flow problem is to determine the maximum value of v.

$$\text{Maximize} \quad v$$
$$\text{subject to} \quad \begin{cases} f_{s1} + f_{s2} - v & = 0, \\ f_{1t} + f_{12} - f_{21} - f_{s1} & = 0, \\ f_{21} + f_{2t} - f_{12} - f_{s2} & = 0, \\ v - f_{1t} - f_{2t} & = 0, \end{cases} \tag{2}$$

$$\text{and} \quad \begin{cases} 0 \le f_{s1} \le c_{s1}, \\ 0 \le f_{s2} \le c_{s2}, \\ 0 \le f_{12} \le c_{12}, \\ 0 \le f_{21} \le c_{21}, \\ 0 \le f_{1t} \le c_{1t}, \\ 0 \le f_{2t} \le c_{2t}. \end{cases} \tag{3}$$

The coefficient matrix on the L.H.S. of equations (2) is simply the incidence matrix of this directed graph.

For a general network $N = (V, E)$, constraints (2) and (3) becomes

$$\sum_{j \in V} f_{ij} - \sum_{j \in V} f_{ji} = \begin{cases} v, & i = s \\ 0, & i \neq s, t \\ -v, & i = t \end{cases} \tag{4}$$

$$0 \leq f_{ij} \leq c_{ij}, \quad \forall (i, j) \in E \tag{5}$$

Any set of numbers $\{f_{ij}\}$ satisfying (4) and (5) is said to be a *feasible flow*. The value f is called the *value of the flow* and is sometimes denoted by $v(f)$ or simply v.

Mathematically, a *flow*, or more precisely an *s-t flow*, f is a function from E into $\mathbb{R}^+$ such that

$$0 \leq f_{ij} \leq c_{ij}, \quad \forall (i, j) \in E$$

and

$$\sum_{\{j|(i,j) \in E\}} f_{ij} = \sum_{\{j|(i,j) \in E\}} f_{ji}, \quad \forall\, i \in V,\ i \neq s, t.$$

For simplicity, given two subsets S and T of V and an s-t flow f from E into $\mathbb{R}^+$, we use (S, T) to denote the set $\{(i,j) \in E | i \in S, j \in T\}$ and

$$f(S, T) \equiv \sum_{(i,j)\in(S,T)} f_{ij}.$$

If S equals to a singleton set $\{i\}$, we write $f(\{i\}, T) = f(i, T)$. In particular, $f(i,j) = f_{ij}$. In this notation, conservation of flows (4) become

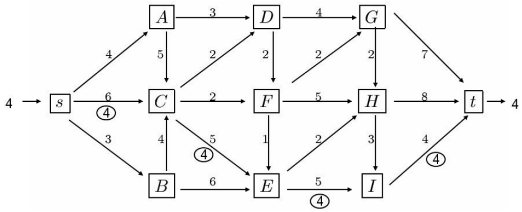
$$f(i, V) - f(V, i) = \begin{cases} v(f), & i = s \\ 0, & i \neq s, t \\ -v(f), & i = t \end{cases} \qquad (6)$$

where the value of the flow is given by

$$v(f) = f(s, V) - f(V, s) = f(V, t) - f(t, V). \qquad (7)$$

Example

Consider the network below where the numbers on the edges represent the capacities.



An s-t flow of value 4 is drawn on the figures where the flow value is marked by circles. Note that for examples $f(C, V) = f(V, C) = 4$ and $f(D, V) = f(V, D) = 0$. Also $f(s, V) = 4 = f(V, t)$ whereas $f(V, s) = f(t, V) = 0$. $\square$

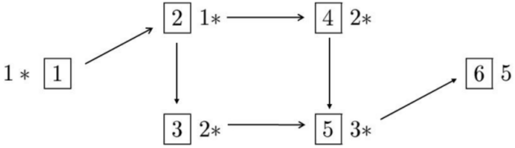
Methods for Maximal-Flow Problems

First-label-first-scan Methods

- To find a nearest path from a source s to a sink t.
- At each step of the procedure, every vertex $i \in V$ is either:
  i) unlabeled (indicated by blank)
  ii) labeled not scanned (indicated by a label $\ell(i)$)
  iii) labeled and scanned ($\ell(i)$ followed by an $*$)
- **First-label-first-scan Method:**
  (1) Label vertex s by $\ell(s) = s$.
  (2) If vertex t is labeled, an s-t path is obtained by tracing backward from t to s using the labels on the vertices; otherwise go to Step 3.
  (3) If all labeled vertices are scanned, there exists no s-t path; otherwise go to Step 4.
  (4) Pick the first labeled but unscanned vertex i, label each unlabeled vertex j such that (i, j) is an edge by $\ell(j) = i$. Indicate vertex i as scanned and return to Step 2.

Example

Consider the network:



An s-t path is $6 \leftarrow 5 \leftarrow 3 \leftarrow 2 \leftarrow 1$. $\square$

Flow Augmenting Path Algorithm for Maximal Flow:

Step 1 Find a s-t path with strictly positive flow capacity for each edge in the path. If no such path exists, we are done.

Step 2 Search this path for the edge with the smallest flow capacity, say c^*, and increase the flow in this path by c^*.

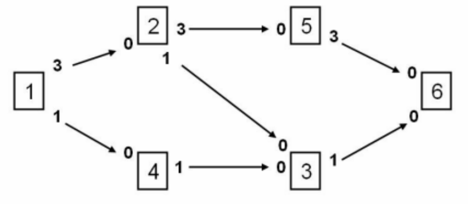
Step 3 Decrease by c^* the flow capacity for each edge in this path.

Step 4 Increase by c^* the flow capacity in the opposite direction for each edge in the path.

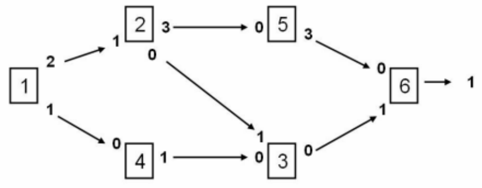
Step 5 Go back to Step 1.

Example

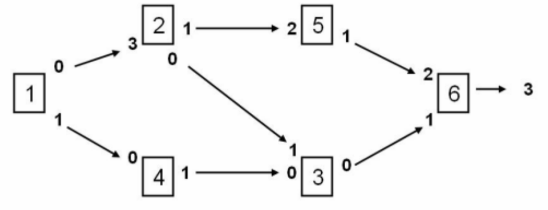
Consider the following network where the numbers on the edges represent the current flow capacities for the forward and the backward directions.



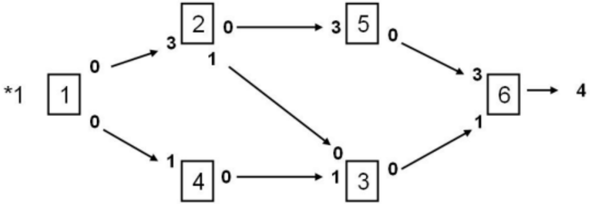
Initially the flow $v = 0$; Augmenting path is $1 \to 2 \to 3 \to 6$ with $c^* = 1$.



$v = 1$; Augmenting path is $1 \to 2 \to 5 \to 6$ with $c^* = 2$.



$v = 3$; Augmenting path is $1 \to 4 \to 3 \to 2 \to 5 \to 6$ with $c^* = 1$. Notice that edge $(3, 2)$ is a *backward* edge.



$v^* = 4$; there is no more augmenting paths. Thus the maximal flow f^* is given by $f(1,2) = 3$, $f(1,4) = 1$, $f(2,3) = 0$, $f(2,5) = 3$, $f(3,6) = 1$, $f(4,3) = 1$, $f(5,6) = 3$. □

Definition

Let P be an *undirected* path from s to t. An edge (i,j) on P is said to be a *forward edge* if it is directed from s to t and *backward edge* otherwise. P is said to be a *flow augmenting path* with respect to a given flow f if

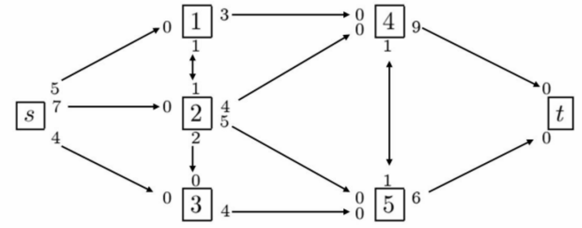
(1) $f(i,j) < c_{ij}$ for each *forward* edge (i,j) on P, and

(2) $f(i,j) > 0$ for each *backward* edge (i,j) on P.

Thus the path $1 \to 4 \to 3 \to 2 \to 5 \to 6$ in the last iteration is a flow augmenting path where $3 \to 2$ is a backward edge.

Exercise

Consider the network:



Maximal Flow and Minimal Cut

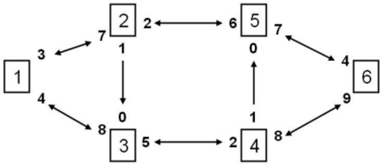
Definition

Given a network $N = (V, E)$ with source s and sink t. Let X and $\bar{X}$ be two non-empty subsets of V such that $X \cap \bar{X} = \phi$ and $X \cup \bar{X} = V$. If $s \in X$ and $t \in \bar{X}$, then $(X, \bar{X})$ is called an *s-t cut* (or simply a *cut*) of the network N. The *capacity* of a cut $(X, \bar{X})$, denoted by $C(X, \bar{X})$, is the sum of the capacities of those edges directed from a vertex in X to a vertex in $\bar{X}$, i.e.

$$C(X, \bar{X}) = \sum_{(i,j)\in(X,\bar{X})} c_{ij}.$$

Example

Consider the following network with capacities listed at the corner of the vertices:



$(X, \bar{X})$	$C(X, \bar{X})$
$(\{1\}, \{2,3,4,5,6\})$	$3+4=7$
$(\{1,2,3,4,5\}, \{6\})$	$7+8=15$
$(\{1,2,3\}, \{4,5,6\})$	$2+5=7$
$(\{1,3,5\}, \{2,4,6\})$	$3+5+6+7=21$
$(\{1,2,3,4\}, \{5,6\})$	$2+1+8=11$

Lemma (1)

Let f be an s-t flow and $(X, \bar{X})$ an s-t cut, then

$$v(f) = f(X, \bar{X}) - f(\bar{X}, X) = \textit{net flow across the s-t cut.}$$

Proof.

We have by (6) and (7)

$$\begin{aligned} v(f) &= f(s, V) - f(V, s) \\ &= f(s, V) - f(V, s) + \sum_{i \in X, i \neq s} [f(i, V) - f(V, i)] \\ &= f(X, V) - f(V, X) \\ &= f(X, \bar{X}) + f(X, X) - f(X, X) - f(\bar{X}, X). \end{aligned}$$

□

Lemma (2)

Given any s-t flow f and s-t cut $(X, \bar{X})$, *we have* $v(f) \le C(X, \bar{X})$. *In particular, we have*

$$\max_f v(f) \le \min_{(X,\bar{X})} C(X, \bar{X}).$$

Proof.

Since $f(X, \bar{X}) \le C(X, \bar{X})$ and $f(\bar{X}, X) \ge 0$, we have

$$v(f) = f(X, \bar{X}) - f(\bar{X}, X) \le C(X, \bar{X}).$$

□

Theorem (1)

(Augmentation Algorithm) An s-t flow f is a maximal flow if and only if it admits no flow augmenting path from s to t.

Proof.

If an augmenting path exists, the current flow is clearly not a maximal flow.
Now suppose f does not admit an augmenting path from s to t. Let X be the set of vertices $\{i\}$ including s for which there is an augmenting path from s to i and $\bar{X}$ be the complementary set of vertices, i.e. $\bar{X} = V \setminus X$.
We claim that for all $i \in X$ and $j \in \bar{X}$, we have $f(i,j) = c_{ij}$ and $f(j,i) = 0$.
For if $f(i,j) < c_{ij}$, obviously we are allowed to flow from i to j, and hence there will be an augmenting path from s to j. If $f(j,i) > 0$, that means we have previously flow from j to i. Now we can form an augmenting path from s to j by first going to i and then augmenting that with a backward edge from i to j.
Hence in both cases, we have an augmenting path from s to j, a contradiction to the fact that $j \in \bar{X}$. □

Proof (con't).

Since $(X, \bar{X})$ is an s-t cut, we have by Lemma (1),

$$\begin{aligned} v(f) &= f(X, \bar{X}) - f(\bar{X}, X) = \\ &= \sum_{i \in X, j \in \bar{X}} f(i,j) - \sum_{j \in \bar{X}, i \in X} f(j,i) \\ &= \sum_{i \in X, j \in \bar{X}} c(i,j) = C(X, \bar{X}), \end{aligned}$$

i.e. f is a maximal flow. □

Theorem (2)

(The Max-flow Min-cut Theorem) For any network the maximal flow value from vertex s to vertex t is equal to the minimal cut capacity, i.e.

$$\max_{f} v(f) = \min_{(X,\bar{X})} C(X,\bar{X}) \, .$$

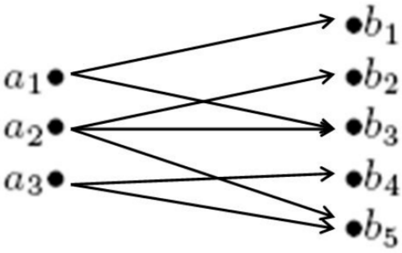
Proof.

A unique minimal cut with respect to the given maximal flow is constructed in the proof of the Theorem (1). □

Example

System of Distinct Representative

Five Senators b_1, b_2, b_3, b_4, b_5 are members of three committees a_1, a_2 and a_3. The membership is as follows:



One member from each committee is to be represented in a super-committee. Is it possible to send one distinct representative from each of the committees?

LP Interpretation of Max-flow Min-cut Problem

- Prove the max-flow min-cut theorem again by using the duality theorems of LP problems.
- The LP for maximal flow problem can be stated as:

$$(P) \quad \begin{array}{ll} \text{Max} & v = f(t,s) \\ \text{s.t.} & \begin{cases} f(i,V) - f(V,i) = 0, & \forall i \in V, \\ 0 \le f(i,j) \le c_{ij}, & \forall (i,j) \in E. \end{cases} \end{array} \tag{8}$$

Notice that there are $|V|$'s conservation constraints and $|E|$'s capacity constraints.

Let us write the cost vector of the primal problem in (9) as $\mathbf{c}^T = (0, \ldots, 0, 1)$, the right hand side vector as $\mathbf{b}^T = (0, \ldots, 0 | \ldots, c_{ij}, \ldots)$ and the solution vector as $\mathbf{x}^T = (\ldots, f_{ij}, \ldots, f_{ts})$. Then we can write the coefficient matrix of the primal in the form:

	(i,j)	(t,s)
s $\vdots$ t	incidence matrix	-1 0 $\vdots$ $+1$
(i,j)	I	0 0 $\vdots$ 0

Lemma (3)

The coefficient matrix A of the maximal flow problem is unimodular.

Proof.

Partition $A = \begin{bmatrix} B \\ C \end{bmatrix}$, where B corresponds to the vertex constraints and C corresponds to the edge constraints. Consider any k-by-k submatrices M_k of A. First we consider the case where M_k is a submatrix of B only. Then there are three cases: (i) all columns of M_k consist of two nonzero entries, (ii) there is a column of M_k consisting of all zero entries, and (iii) there is a column of M_k consisting of only one nonzero entries. In case (i), then the two nonzero entries must be 1 and -1. Hence if we sum all the rows in M_k, we have a zero vector. Hence M_k is singular and therefore $\det M_k = 0$. In case (ii), of course M_k is singular and therefore again $\det M_k = 0$. In case (iii), then we can expand the determinant at the only nonzero entry in that column and get $\det M_k = \pm \det M_{k-1}$. By repeating the arguments, we see that the conclusion of the Lemma is valid. □

Proof. (con't).

Now suppose $M_k = \begin{bmatrix} B_k \\ C_k \end{bmatrix}$, where B_k and C_k are submatrices of B and C respectively. If any one of the rows of C_k is zero, then $\det(M_k) = 0$, and we are done. If one of the rows of C_k is nonzero, then because of the form of C (which is an identity matrix plus a zero column), the nonzero row must contain at most one nonzero entry and the nonzero entry must be 1. Expanding the determinant of M_k at that entry and we have $\det(M_k) = \det(M_{k-1})$, where M_{k-1} is a square submatrix of M_k. Now the proof can be completed by recursion as M_{k-1} is just an $(k-1)$-by-$(k-1)$ submatrix of A. □

Prime and Dual Problem

$$
(Prime) \quad \begin{array}{ll} \text{Max} & v = f(t,s) \\ \text{s.t.} & \begin{cases} f(i,V) - f(V,i) = 0, & \forall i \in V, \\ 0 \le f(i,j) \le c_{ij}, & \forall (i,j) \in E. \end{cases} \end{array} \tag{9}
$$

$$
(Dual) \quad \begin{array}{ll} \text{Min} & \sum_{(i,j)\in E} c_{ij} w_{ij} \\ \text{subject to} & \begin{cases} u_i - u_j + w_{ij} \ge 0, & (i,j) \in E, \\ u_t - u_s \ge 1, & \\ u_i \text{ unrestricted}, & i \in V, \\ w_{ij} \ge 0, & (i,j) \in E. \end{cases} \end{array} \tag{10}
$$

Lemma (4)

For every s-t cut $(X, \bar{X})$, there exists a feasible solution $(\mathbf{u}, \mathbf{w})$ to the dual with the objective function value being equal to $C(X, \bar{X})$.

Proof.

Set

$$u_i = \begin{cases} 0, & i \in X \\ 1, & i \in \bar{X} \end{cases} \quad \text{and} \quad w_{ij} = \begin{cases} 1, & (i,j) \in (X, \bar{X}), \\ 0, & (i,j) \notin (X, \bar{X}). \end{cases}$$

We claim that $(\mathbf{u}, \mathbf{w})$ is feasible, i.e. it satisfies (10). In fact, we can check all four possible cases where i and j are either in X or $\bar{X}$. For example, if $i \in X$ and $j \in \bar{X}$, then $u_i - u_j + w_{i,j} = 0 - 1 + 1 = 0$. Since $u_t - u_s = 1 - 0 = 1$, the last constraint is also satisfied. Finally

$$C(X, \bar{X}) = \sum_{(i,j)\in(X,\bar{X})} c_{ij} = \sum_{(i,j)\in(X,\bar{X})} c_{ij} w_{ij} = \sum_{(i,j)\in E} c_{ij} w_{ij}.$$

□

Corollary

Given any s-t flow f and any s-t cut $(X, \bar{X})$,

$$v(f) \leq C(X, \bar{X}).$$

Proof.

If $(X, \bar{X})$ is an s-t cut, then there exists a feasible solution to the dual with the objective function value being equal to $C(X, \bar{X})$. By the weak duality of LP, we have

$$C(X, \bar{X}) = \sum_{(i,j)\in E} c_{ij} w_{ij} \geq f(t, s) = v(f).$$

□

Lemma (5)

For every BFS $(\mathbf{u}, \mathbf{w})$ *to the dual, there exists an s-t cut* $(X, \bar{X})$ *such that*

$$C(X, \bar{X}) \leq \sum_{(i,j)\in E} c_{ij} w_{ij}. \tag{11}$$

Proof.

Since $\det M = \det M^T$ and the coefficient matrix A for the primal is totally unimodular, we see that the coefficient matrix A^T for the dual is also totally unimodular. Hence every BFS to the dual is integer-valued. In particular, if $w_{ij} > 0$, then $w_{ij} \geq 1$. Given an s-t path, if we sum over the dual constraints over the path, we get

$$(u_s - u_t) + \sum_{(i,j)\in s\text{-}t \text{ path}} w_{ij} \geq 0.$$

Since $u_t - u_s \geq 1$, we have $\sum_{(i,j)\in s\text{-}t \text{ path}} w_{ij} \geq 1$. By the integral and non-negativity properties of $\mathbf{w}$, there exists at least one edge (k, ℓ) in the path such that $w_{k\ell} \geq 1$. $\square$

Proof. (con't).

Let

$$X \equiv \{s\} \cup \{k \mid \text{there exists a path from } s \text{ to } k \text{ along edges with } w_{ij} = 0\}.$$

Let $\bar{X} \equiv V \setminus X$. Since there is some $w_{k\ell} \geq 1$ on every s-t path, $t \in \bar{X}$. Hence $(X, \bar{X})$ is an s-t cut and $w_{ij} \geq 1$ if $(i,j) \in (X, \bar{X})$. Thus

$$\sum_{(i,j)\in E} c_{ij} w_{ij} \geq \sum_{(i,j)\in(X,\bar{X})} c_{ij} w_{ij} \geq \sum_{(i,j)\in(X,\bar{X})} c_{ij} = C(X, \bar{X}).$$

□

Corollary

For an optimal solution $(\mathbf{u}^*, \mathbf{w}^*)$*, there exists an s-t cut that satisfies*

$$C(X, \bar{X}) = \sum_{(i,j)\in E} c_{ij} w_{ij}^*. \tag{12}$$

Proof.

Assume that $(\mathbf{u}^*, \mathbf{w}^*)$ is an optimal basic feasible solution. Let $(X^*, \bar{X}^*)$ be the s-t cut corresponding to $(\mathbf{u}^*, \mathbf{w}^*)$, i.e. $C(X^*, \bar{X}^*) \leq \sum_{(i,j)\in E} c_{ij} w_{ij}^*$ by (11). By Lemma 10, given this s-t cut, there exists a feasible solution $(\hat{\mathbf{u}}, \hat{\mathbf{w}})$ to the dual such that

$$C(X^*, \bar{X}^*) = \sum_{(i,j)\in E} c_{ij} \hat{w}_{ij}.$$

Since $(\mathbf{u}^*, \mathbf{w}^*)$ is optimal, we then have

$$\sum_{(i,j)\in E} c_{ij} w_{ij}^* \leq \sum_{(i,j)\in E} c_{ij} \hat{w}_{ij} = C(X^*, \bar{X}^*) \leq \sum_{(i,j)\in E} c_{ij} w_{ij}^*$$

where the last inequality follows from (11). Thus (12) holds. □

Theorem (3)

(Max-flow Min-cut via LP duality)

$$\max_f v(f) = \min_{(X,\bar{X})} C(X,\bar{X}) .$$

Proof.

We have by the strong duality theorem,

$$v^*(f) = f^*(t,s) = \sum_{(i,j)\in E} c_{ij} w^*_{ij} = C^*(X,\bar{X}).$$

Note that the cut $(X,\bar{X})$ has to be minimum. In fact, if there exists another cut $(Y,\bar{Y})$ such that $C(Y,\bar{Y}) < C(X,\bar{X})$, then by Lemma 10, there exists a feasible solution $(\tilde{\mathbf{u}},\tilde{\mathbf{w}})$ with $\sum_{(i,j)\in E} c_{ij}\tilde{w}_{ij} = C(Y,\bar{Y}) < C(X,\bar{X}) = \sum_{(i,j)\in E} c_{ij} w^*_{ij}$, a contradiction to the optimality of $(\mathbf{u}^*,\mathbf{w}^*)$. □