Week 9

MATH 2040

November 17, 2020

1 Review

1. **Definition:** T-invariant: Let $T : V \to V$ be a linear transformation, a subspace $W \subset V$ is called T-invariant if $T(W) \subset W$.
2. **Example:** $\{0\}, V, N(T)$ and $R(T)$ are T-invariant. Since $T(0) = 0$, $T(V) \subset V$, $T(N(T)) = \{0\} \subset N(T)$ and $T(R(T)) \subset T(V) = R(T)$
3. **Fact:**
   (a) f_T is divisbile by $f_{T|_W}$.
   (b) Eigenvalues and eigenvectors of $T|_W$ are also eigenvalues and eigenvectors of T.
   (c) T is diagonalizable $\Rightarrow T|_W$ is diagonalizable.

2 Problems

1. Let $W \subset V$ be T-invariant and $\lambda_1, \cdots, \lambda_k$ be distinct eigenvalues of T. Suppose there are $v_1, \cdots, v_k \in V$ such that $v_i \in E_{\lambda_i}(T)$ and $v_1 + \cdots + v_k \in W$, show that $v_i \in W$ for all i.

   Ans:

   Method 1: Remark the hypothsis that $v_1 + \cdots + v_k \in W \Rightarrow v_i \in W$ for all i as $P(k)$.

   For $k = 1$, $P(1)$ holds obviously.

   For $k = n$, we assume that $P(n-1)$ holds, then since W is T-invariant and $v_1 + \cdots + v_n \in W$, we have that $T(v_1 + \cdots + v_n) \in W$ and $\lambda_n(v_1 + \cdots + v_n) \in W$. Thereforce,

   $$T(v_1 + \cdots + v_k) - \lambda_n(v_1 + \cdots + v_k) = (\lambda_1 - \lambda_n)v_1 + \cdots + (\lambda_{n-1} - \lambda_n)v_{n-1} \in W.$$

   Since $(\lambda_i - \lambda_n)v_i \in E_{\lambda_i}(T)$, by induction hypothsis we have that $(\lambda_i - \lambda_n)v_i \in W$. $\lambda_i - \lambda_n \neq 0$ so $v_i \in W$ for all $i = 1, \cdots, n-1$. Thereforce $v_n = (v_1 + \cdots + v_n) - v_1 - \cdots - v_{n-1} \in W$.

   Method 2: There is a polynomial $f_i \in P_{k-1}(\mathbb{F})$ such that

   $$f_i(\lambda_j) = \begin{cases} 1, \text{if } i = j \\ 0, \text{if } i \neq j \end{cases}$$

   (See Tut7 Q5). Suppose that $f_j(t) = \sum_{l=1}^{k-1} c_l t^l$ Then we have that

   $$f_i(T)(v_j) = \sum_{l=1}^{k-1} c_l T^l(v_j) = \sum_{l=1}^{k-1} c_l \lambda_j^l v_j = f_i(\lambda_j)v_j = \begin{cases} v_i, \text{if } i = j \\ 0, \text{if } i \neq j \end{cases}$$

   Therefore, $f_i(T)(v) = v_i$ for all $i = 1, \cdots, k$. Since $v \in W$ and W is T-invariant, so $T^l(v) \in W$ for all l, then $v_i = f_i(T) = \sum_{l=1}^{k-1} c_l T^l(v) \in W$.

2. Given an example of that $T : V \to V$ is linear and subspace $W \subset V$ is T-invariant such that $T|_W$ is diagonalizable but T not and $\dim W = \dim V - 1$.

   Ans: Let $V = \mathbb{R}^2$, $T(x, y) = (x + y, y)$, $W = \{(0, y) \in \mathbb{R}^2 | y \in R\}$, it is easy to check $T|_W$ is diagonalizable but T not.

3. Let $A \in M_{n\times n}(\mathbb{F})$ be an upper trianglar matrix such that all diagonal entries of A are the same. Suppose A is diagonalizablemm show that A is diagonal.

Ans: Suppose that $A = \begin{pmatrix} a & * & * & \cdots & * \\ 0 & a & * & \cdots & * \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & a & * \\ 0 & \cdots & 0 & 0 & a \end{pmatrix}$, then $f_A(t) = (t-c)^n$, which means the eigenvalue of A can only be c. Since A is diagonalizable, there exists some invertable matrix Q such that $Q^{-1}AQ = cI$, so $A = Q(cI)Q^{-1} = cI$ and A is diagonal.

4. Let $T : V \to V$ be linear and $W \subset V$ be T-invariant. For $u, v \in V$, we say $u \sim v$ if $u - v \in W$. Denote $[v] = \{u \in V | u \sim v\}$ and $V/W = \{[v] | v \in V\}$, then define addition and multiple on V/W as

$$[u] + [v] = [u + v], \alpha[u] = [\alpha u].$$

Show that

(a) $\overline{T}[v] = [T(v)]$ defines a linear transformation $\overline{T} : V/W \to V/W$.

(b) T is diagonalizable $\Rightarrow \overline{T}$ is diagonalizable.

(c) $f_T = f_{T|_W} \cdot f_{\overline{T}}$.

Ans:

(a) i. $\overline{T}([0]) = [T(0)] = [0]$

ii. $\overline{T}(a[u] + [v]) = [T(au + v)] = [aT(u) + T(v)] = a[T(u)] + [T(v)] = a\overline{T}([u]) + \overline{T}([v])$.

Thereforce, $\overline{T}$ is linear.

(b) Since T is diagonalizable, $\forall v \in W$, $v = v_1 + \cdots + v_k$, where $v_i \in E_{\lambda_i}$, $T(v_i) = \lambda_i v_i$. Then $\forall [v] \in V/W$, $[v] = [v_1] + \cdots + [v_k]$,

$$\overline{T}([v]) = \overline{T}([v_1]) + \cdots + \overline{T}([v_k]) = [T(v_1)] + \cdots + [T(v_k)] = \lambda_1[v_1] + \cdots + \lambda_k[v_k],$$

which means $\overline{T}$ is also diagonalizable.

(c) Let $\beta_1 = \{w_1, \cdots, w_n\}$ be the basis of W, $\gamma = \{w_1, \cdots, w_n, v_1, \cdots, v_r\}$ be the basis of V and $\beta_2 = \{v_1, \cdots, v_r\}$. Claims that $\{[v_1], \cdots, [v_r]\}$ is the basis of V/W. Suppose $a_1[v_1] + \cdots + a_r[v_r] = 0$, then $a_1v_1 + \cdots + a_rv_r = w \in W$. w can be written as $w = c_1w_1 + \cdots + c_nw_n$, so we have $a_1v_1 + \cdots a_rv_r - c_1w_1 - \cdots - c_nw_n = 0$. Since γ is basis of V, $a_i = 0$ for all i, the claim is true. Let $U = \text{span}(\beta_2)$, $[T|_W]_{\beta_1} = A \in M_{n \times n}(\mathbb{F})$, $[T|_U]_{\beta_2} = B \in M_{r \times r}(\mathbb{F})$, then we have $[T]_\gamma = \begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$ since W is T-invariant. So $f_T = f_{T|_W} \cdot f_{T|_U} = f_{T|_W} \cdot f_{\overline{T}}$.