Week 6

MATH 2040B

November 4, 2020

1 Concepts

1. Inverible: $T : V \to W$ is inverible if T is bijective and there exists $T^{-1} : W \to V$ such that $T \circ T^{-1} = I_W$ and $T^{-1} \circ T = I_V$.

2. T^{-1} is linear if T is linear.

3. $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.

4. V is isomorphic to W if there exists an inverible linear transformation $T : V \to W$ and such T is called an isomorphism from V to W.

5. V is isomorphic to W iff $\dim(V) = \dim(W)$.

6. Standard representation: β is ordered basis for an n-dimension vector space V over $\mathbb{F}$, then the $\Phi_\beta : V \to \mathbb{F}^n, x \to [x]_\beta$ is called standard representation of V with respect to β.

7. Change of coordinate matrix: The matrix $Q = [I_v]_{\beta'}^\beta$ is called the change of coordinate matrix from β' to β, where β and β' are ordered basis of finite dimension vector space V, and for all $v \in V$, $[v]_\beta = Q[v]_{\beta'}$.

2 Problems

1. Let $f : U_1 \to U_2$ be a linear transformation, then for any vector space W, there exists a linear transformation

$$f_* : \mathcal{L}(W, U_1) \to \mathcal{L}(W, U_2), \alpha \to f \circ \alpha.$$

If we have another linear transformations $g : U_2 \to U_3$ and corresponding g_* such that

$$U_1 \xrightarrow{f} U_2 \xrightarrow{g} U_3$$
$$\mathcal{L}(W, U_1) \xrightarrow{f_*} \mathcal{L}(W, U_2) \xrightarrow{g_*} \mathcal{L}(W, U_3),$$

then prove that

$$R(f) \subset N(g) \Rightarrow R(f_*) \subset N(g_*)$$
$$R(f) \supset N(g) \Rightarrow R(f_*) \supset N(g_*)$$

Ans:

$\subset$: Note that $R(f) \subset N(g)$ would imply $g \circ f = 0$. For all $\alpha \in R(f_*)$, i.e. $\alpha = f_*(\beta) = f \circ \beta$, where $\beta \in \mathcal{L}(W, U_1)$, then we have $g_*(\alpha) = g \circ \alpha = g \circ f \circ \beta = 0$, $R(f_*) \subset N(g_*)$.

$\supset$: For all $\alpha \in N(g_*)$, we have $g_*(\alpha) = g \circ \alpha = 0$. This would say that $R(\alpha) \subset N(g) \subset R(f)$. Let $S = \{s_1, s_2, \cdots, s_n\}$ be a basis of W, since $R(\alpha) \subset R(f)$, we have that for all i, there exists some $t_i \in U_1$ such that $\alpha(s_i) = f(t_i)$. Define that $\beta : W \to U_1$, $\beta(s_i) = t_i$ and it is easy to check β is well-defined and linear. For all $i = 1, 2, \cdots, n$, $f \circ \beta(s_i) = f(t_i) = \alpha(s_i)$, so $f \circ \beta = \alpha$, which means $\alpha \in R(f_*)$ and therefore $N(g_*) \subset R(f_*)$.

2. Let $f : U_1 \to U_2$ be a linear transformation, then for any vector space W, there exists a linear transformation

$$f_* : \mathcal{L}(U_2, W) \to \mathcal{L}(U_1, W), \alpha \to \alpha \circ f.$$

If we have another linear transformations $g : U_2 \to U_3$ and corresponding g_* such that

$$U_1 \xrightarrow{f} U_2 \xrightarrow{g} U_3$$
$$\mathcal{L}(U_3, W) \xrightarrow{g_*} \mathcal{L}(U_2, W) \xrightarrow{f_*} \mathcal{L}(U_1, W),$$

then prove that

$$R(f) \subset N(g) \Rightarrow R(f_*) \subset N(g_*)$$
$$R(f) \supset N(g) \Rightarrow R(f_*) \supset N(g_*)$$

Ans:

$\subset$: Note that $R(f) \subset N(g)$ would imply $g \circ f = 0$. For all $\alpha \in R(g_*)$, i.e. $\alpha = g_*(\beta) = \beta \circ g$, where $\beta \in \mathcal{L}(W, U_1)$, then we have $f_*(\alpha) = \alpha \circ f = \beta \circ g \circ f = 0$, $R(g_*) \subset N(f_*)$.

$\supset$: For all $\alpha \in N(f_*)$, we have $f_*(\alpha) = \alpha \circ f = 0$. This would say that $N(g) \subset R(f) \subset N(\alpha)$. The proof of Rank-Nullity Theorem tells us that there exists a basis $\{r_1, r_2, \cdots, r_k, s_1, s_2, \cdots, s_l\}$ of U_2 such that $\{r_1, r_2, \cdots, r_k\}$ is a basis of $N(g)$ and $\{g(s_1), g(s_2), \cdots, g(s_l)\}$ is linear independent in U_3. Then we can extend the vectors $\{g(s_1), g(s_2), \cdots, g(s_l)\}$ to a basis $\{g(s_1), g(s_2), \cdots, g(s_l), t_1, t_2, \cdots, t_p\}$ of U_2. Define that $\beta : U_3 \to W$, $\beta(g(s_i)) = \alpha(s_i)$, $\beta(t_j) = 0$ and it is easy to check β is well-defined and linear. By the defination, $\beta \circ g(s_i) = \alpha(s_i)$ and $\beta \circ g(r_i) = \beta(0) = 0 = \alpha(0)$, so $\beta \circ g = \alpha$, which means $\alpha \in R(g_*)$ and therefore $N(f_*) \subset R(g_*)$.

3. Show that for all $c_0, c_1, \cdots, c_n \in \mathbb{F}$, there exists a polynomial $p \in P_n(\mathbb{F})$ such that

$$p(i) = c_i, i = 0, 1, \cdots n.$$

Ans: Defin a linear transformation $T : P_n(\mathbb{F}) \to \mathbb{F}^{n+1}$ given by

$$T(p) = (p(0), p(1), \cdots, p(n)).$$

Let $p \in N(T)$, then $p(0) = p(1) = \cdots = p(n) = 0$, p has $n+1$ roots. But p is degree n, so we have $p = 0$ and $N(T) = \{0\}$. Therefore $\text{rank}T = \dim P_n(\mathbb{F}) - \dim N(T) = n + 1 = \dim \mathbb{F}^{n+1}$ and T is surjective, so there exists such p.