Week 5

MATH 2040B

October 13, 2020

1 Concepts

1. Matrix representation: Given a linear transformation T from finite dimension vector space U to finite dimension vector space V, $\beta = \{\beta_1, \cdots, \beta_n\}$ is an ordered basis of U and $\gamma = \{\gamma_1, \cdots, \gamma_m\}$ is an ordered basis of V, if $T(\beta_j) = \sum_{i=1}^{m} T_{ij}\gamma_i$, then the $m \times n$ matrix $[T]_\beta^\gamma = (T_{ij})$ is called the matrix representation of T.

2. The coordinate representation corresponing to γ of $T(\beta_j)$ is $[T(\beta_j)]_\gamma = (T_{1j}, \cdots, T_{mj})^T$, and $[T]_\beta^\gamma$ can be writed as the combination of these column vectors $[T]_\beta^\gamma = [[T(\beta_1)]_\gamma, \cdots, [T(\beta_n)]_\gamma]$.

3. If $U = V$ and $\beta = \gamma$, then the matrix representation of T can be simplified as $[T]_\beta$

2 Problems

1. $T : \mathbb{R}^2 \to \mathbb{R}^3$, $T(a, b) = (a - b, a + b, a)$, find $[T]_\beta^\gamma$ where β, γ are standard basis of $\mathbb{R}^2$ and $\mathbb{R}^3$.

   Ans: $T(1, 0) = (1, 1, 1)$ and $T(0, 1) = (-1, 1, 0)$, then

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$$

2. $T : \mathbb{R}^2 \to \mathbb{R}^3$, $T(a,b) = (a-b, a+b, a)$,find $[T]_\beta^\gamma$ where $\beta = \{(1,1),(1,-1)\}$ and $\gamma = \{(0,1,1),(1,0,1),(1,1,0)\}$.

   Ans: $T(1,1) = (0,2,1)$ and $T(1,-1) = (2,0,1)$. Suppose that

$$[T]_\beta^\gamma = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \\ t_{31} & t_{32} \end{pmatrix}$$

   then we have

$$\begin{cases} t_{21} + t_{31} = 0 \\ t_{11} + t_{31} = 2 \\ t_{11} + t_{21} = 1 \\ t_{22} + t_{32} = 2 \\ t_{12} + t_{32} = 0 \\ t_{12} + t_{22} = 1 \end{cases}$$

   Solve above equations we get that $t_{11} = \frac{3}{2}, t_{21} = -\frac{1}{2}, t_{31} = \frac{1}{2}, t_{12} = -\frac{1}{2}, t_{22} = \frac{3}{2}, t_{32} = \frac{1}{2}$, so

$$[T]_\beta^\gamma = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

3. Let $\beta = \{\beta_1, \cdots, \beta_n\}$, $\gamma = \{\gamma_1, \cdots, \gamma_m\}$ be basis of vector space U and V. $\mathcal{L}(U, V)$ is the set of all linear transformation from U to V. It is known that $\mathcal{L}(U, V)$ is a vector space with addition: $(P + Q)(u) = P(u) + Q(u)$ and scalar multiple: $(aP)(u) = aP(u)$. Show that the map $T : \mathcal{L}(U, V) \to M_{m\times n}(F)$ that $T(P) = [P]_\beta^\gamma$ is linear.

Ans: we need to check that $T(P + Q) = T(P) + T(Q)$ and $T(aP) = aT(P)$.

(a) Assume that $T(P) = [P]_\beta^\gamma = (A_{ij})$, $T(Q) = [Q]_\beta^\gamma = (B_{ij})$ and $T(P + Q) = [P + Q]_\beta^\gamma = (C_{ij})$, which is equal to

$$P(\beta_j) = \sum_{i=1}^{m} A_{ij}\gamma_i$$
$$Q(\beta_j) = \sum_{i=1}^{m} B_{ij}\gamma_i$$
$$(P + Q)(\beta_j) = \sum_{i=1}^{m} C_{ij}\gamma_i.$$

On the other side,

$$(P + Q)(\beta_j) = P(\beta_j) + Q(\beta_j) = \sum_{i=1}^{m} A_{ij}\gamma_i + \sum_{i=1}^{m} B_{ij}\gamma_i = \sum_{i=1}^{m}(A_{ij} + B_{ij})\gamma_i$$

Therefore, $C_{ij} = A_{ij} + B_{ij}$ and $T(P + Q) = [P + Q]_\beta^\gamma = [P]_\beta^\gamma + [Q]_\beta^\gamma = T(P) + T(Q)$.

(b) Assume that $T(P) = [P]_\beta^\gamma = (A_{ij})$ and $T(aP) = [aP]_\beta^\gamma = (D_{ij})$, which is equal to

$$P(\beta_j) = \sum_{i=1}^{m} A_{ij}\gamma_i$$
$$(aP)(\beta_j) = \sum_{i=1}^{m} D_{ij}\gamma_i.$$

On the other side,

$$(aP)(\beta_j) = aP(\beta_j) = a\sum_{i=1}^{m} A_{ij}\gamma_i = \sum_{i=1}^{m} aA_{ij}\gamma_i$$

Therefore, $D_{ij} = aA_{ij}$ and $T(aP) = [aP]_\beta^\gamma = a[P]_\beta^\gamma = aT(P)$.

So T is linear.

4. Prove that $T : \mathcal{L}(U, U) \to M_{n\times n}(F)$ is injective.

   Ans: Let $P \in N(T)$, then $[P]_\beta = (P_{ij}) = 0$ and $P_{ij} = 0$ for any $i, j \in \{1, \cdots, n\}$. So we have $P(\beta_j) = \sum_{i=1}^{n} 0 \times \beta_i = 0$, and for any $u \in U$, $u = \sum_{i=1}^{n} a_i\beta_i$, $P(v) = \sum_{i=1}^{n} a_i P(\beta_i) = 0$, so $P = 0$, $N(T) = \{0\}$ and then T is injective.

5. Suppose $P : U \to U$ is linear, show that the following two statements are equivalent.

(1). $P^2 = P$;

(2). For some basis η of U and $r \leq n = \dim(U)$,

$$[P]_\eta = \begin{pmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{pmatrix}$$

Ans: (1) $\Rightarrow$ (2): By R-N theorem, we can assume that $\eta_R = \{u_1, \cdots, u_r\}$ is a basis of $R(P)$ and $\eta_N = \{v_1, \cdots, v_{n-r}\}$ is a basis of $N(P)$. Claims that $\eta = \eta_R \cup \eta_N$ is a basis of U. To prove this, we just need to check η is linearly independent. Suppose

$$a_1u_1 + \cdots + a_ru_r + b_1v_1 + \cdots + b_{n-r}v_{n-r} = 0.$$

Applying P we get that

$$a_1P(u_1) + \cdots + a_rP(u_r) = 0.$$

Note that $u_i \in R(P)$, there exists some $x_i \in U$ such that $P(x_i) = u_i$, so $P(u_i) = P^2(x_i) = P(x_i) = u_i$, then

$$a_1P(u_1) + \cdots + a_rP(u_r) = a_1u_1 + \cdots + a_ru_r = 0,$$

which means $a_1 = \cdots = a_r = 0$. Then

$$a_1u_1 + \cdots + a_ru_r + b_1v_1 + \cdots + b_{n-r}v_{n-r} = b_1v_1 + \cdots + b_{n-r}v_{n-r} = 0.$$

which means $b_1 = \cdots = b_{n-r} = 0$. So η is linearly independent so the claim is proved. Now $P(u_i) = u_i$ and $P(v_i) = 0$, then

$$[P]_\eta = \begin{pmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{pmatrix}$$

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(2) $\Rightarrow$ (1): Since $[P]_\eta = \begin{pmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{pmatrix}$,

$$[P^2]_\eta = [P]_\eta[P]_\eta = \begin{pmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{pmatrix} = [P]_\eta.$$

By the conclusion of Q4, $T : \mathcal{L}(U, U) \to M_{m\times m}(F)$ is injective and so $P^2 = P$.