Week 4

MATH 2040B

October 6, 2020

1 Concepts

1. Vector space
2. Subspace
3. Linear combination and Span
4. Linear dependent and Linear independent
5. Basis and Dimension
6. Replacement theorem
7. Linear transformation
8. Null space and Range
9. Rank and Nullity
10. Rank-nullity theorem

2 Remarks

1. $W_1 \cap W_2$ is vector space, $W_1 \cup W_2$ may not.
2. $\{0\}$ is subspace of any vector space V.
3. $\mathrm{span}(\emptyset) = \{0\}$ and $\emptyset$ is the basis of $\{0\}$.
4. $\mathrm{span}(S)$ is the smallest subspace contains S.
5. Any finite spanning set can be reduced to a basis.
6. Any finite linear independent set can be extended to a basis.
7. Linear transformation preserves linear combination.
8. T is injective $\Leftrightarrow N(T) = \{0\} \Leftrightarrow \mathrm{Nullity}(T) = 0$.
9. T is surjective $\Leftrightarrow R(T) = W \Leftrightarrow \mathrm{Rank}(T) = \dim(W)$.

3 Formula

1. $\#L \leq \dim(V) = \#B \leq \#S$, where V is a vector space, L is any linear independent set, B is any basis and S is any spanning set.
2. $N(T) = \{x \in V : T(x) = 0\}$.
3. $R(T) = \{T(X) : x \in V\}$.
4. $\mathrm{Nullity}(T) = \dim(N(T))$.
5. $\mathrm{Rank}(T) = \dim(R(T))$.
6. $\mathrm{Rank}(T) + \mathrm{Nullity}(T) = \dim(W)$.

4 Class note details

$+\begin{cases} \text{(VS1)}: & \vec{x}+\vec{y} = \vec{y}+\vec{x} \quad \forall \vec{x},\vec{y}\in V \\ \text{(VS2)}: & (\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z}) \quad \forall \vec{x},\vec{y},\vec{z}\in V \\ \text{(VS3)}: & \exists\, \vec{0}\in V \text{ s.t. } \vec{x}+\vec{0}=\vec{x} \quad \forall \vec{x}\in V \\ \text{(VS4)}: & \forall \vec{x}\in V,\ \exists\, \vec{y}\in V \text{ s.t. } \vec{x}+\vec{y}=\vec{0} \quad \text{(inverse)} \end{cases}$

$\cdot\begin{cases} \text{(VS5)}: & \underset{\in F}{1}\,\vec{x} = \vec{x} \quad \forall \vec{x}\in V \\ \text{(VS6)}: & (\underset{\in F}{a}\,\underset{\in F}{b})\vec{x} = a(b\vec{x}) \quad \forall a,b\in F,\ \forall \vec{x}\in V \end{cases}$

$+/\cdot\begin{cases} \text{(VS7)}: & \underset{\in F}{a}(\underset{\in V}{\vec{x}}+\underset{\in V}{\vec{y}}) = a\vec{x}+a\vec{y} \quad \forall a\in F,\ \forall \vec{x},\vec{y}\in V \\ \text{(VS8)}: & (a+b)\vec{x} = a\vec{x}+b\vec{x} \quad \forall a,b\in F,\ \forall \vec{x}\in V \end{cases}$

Remark: an element in F is called scalar.
" " " V is called vector.

Subspace

Definition: A subset W of a vector space V over a field F is called a subspace of V if W is a vector space over F under the same addition and scalar multiplication inherited from V.

Proposition: Let V be a vector space over F. A subset $W\subset V$ is a subspace iff the following 3 conditions holds:

(a) $\vec{0}_V \in W$

(b) $\vec{x}+\vec{y}\in W$, $\forall \vec{x},\vec{y}\in W$ (closed under addition)

(c) $a\vec{x}\in W$, $\forall a\in F, \vec{x}\in W$ (closed under scalar multiplication)

Lecture 3:

Recall: • Linear combination of S :

$$\vec{v} = \underset{\hat{F}}{a_1}\underset{\hat{S}}{\vec{v}_1} + \underset{\hat{F}}{a_2}\underset{\hat{S}}{\vec{v}_2} + \dots + \underset{\hat{F}}{a_n}\underset{\hat{S}}{\vec{v}_n}$$

• a_i's are called coefficients

• $\mathrm{Span}(S) = \{ a_1\vec{v}_1 + \dots + a_n\vec{v}_n : a_i \in F,\ i=1,2,\dots,n,\ n \in \mathbb{N},\ \vec{v}_j \in S \}$

Lecture 4:

Recall: 1. Linearly independent means NOT linearly dependent.

Linearly dependent S,

$\exists$ distinct $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n \in S$ and $\exists\ a_1, a_2, \dots, a_n \in F$ (not all zero)

such that:

$$a_1\vec{u}_1 + a_2\vec{u}_2 + \dots + a_n\vec{u}_n = \vec{0}.$$

2. Linearly indepedent S

$\Leftrightarrow$ Each $\vec{x} \in \mathrm{Span}(S)$ can be expressed in a unique way as lin. comb. of S.

$\Leftrightarrow$ $\vec{0} = a_1\underset{\hat{S}}{\vec{u}_1} + \dots + a_n\underset{\hat{S}}{\vec{u}_n} \Rightarrow a_1 = a_2 = \dots = a_n = 0.$

Definition: A basis for a vector space V is a subset $\beta \subset V$ such that:

• β is linearly independent and

• β spans V, i.e. $\mathrm{Span}(\beta) = V$.

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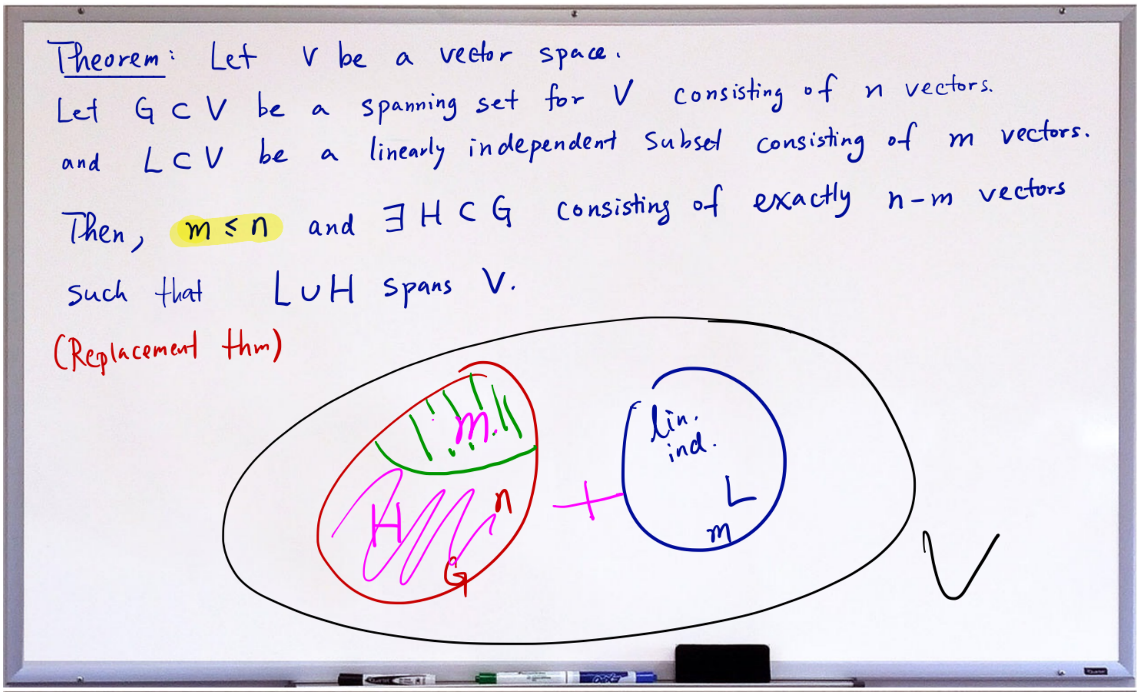

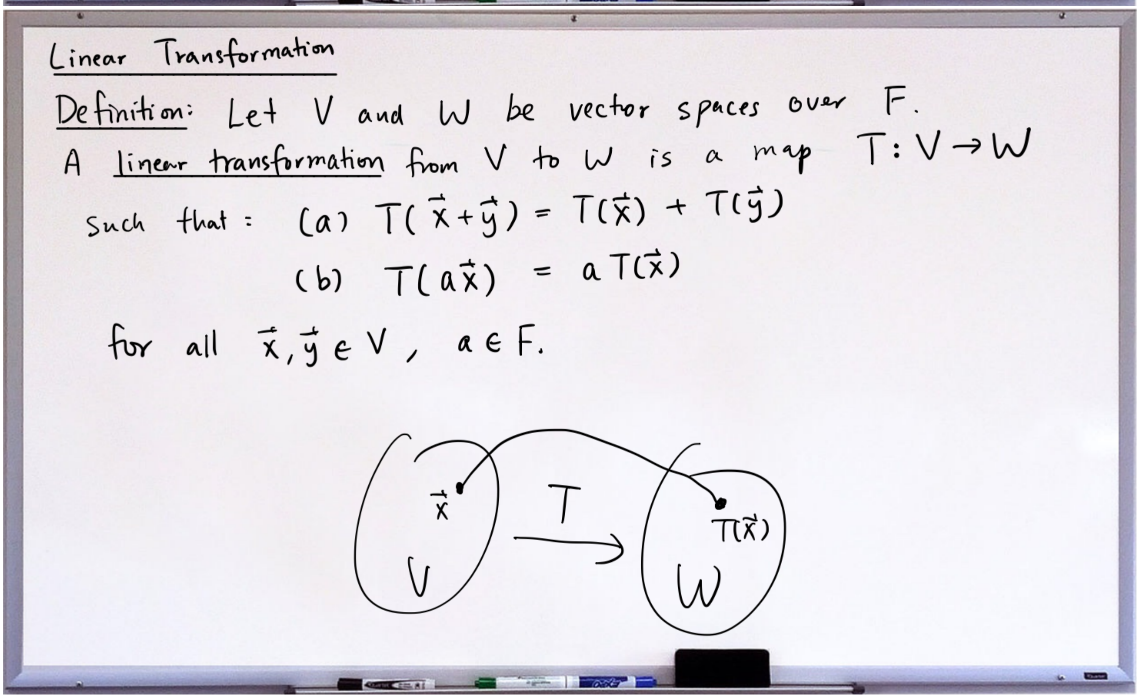

5 Problems

1. $V = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$, then find a basis of V.

2. Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $N(T) = R(T)$.

3. Determine whether the following sets S are linear independent or not. If not, reduce it to a linear independent set β such that $\text{span}(\beta) = \text{span}(S)$. If so, determine whether it is the basis of vector space V.

   (a) $S = \{x^2 + x^5, -2x^2 + x^{10}, 3x^2 - x^5\}, V = P_{10}(\mathbb{R})$,

   (b) $S = \{(2, 3, 4), (1, 3, 5), (5, 9, 15), (4, 3, 2)\}, V = \mathbb{R}^3$.

4. Let V be a finite-dimension vector space over $\mathbb{R}$ and Z is a subspace of V, for any $v \in V$,

$$[v] = \{u \in V : v - u \in Z\}$$

   Let $V/Z = \{[u] : u \in V\}$. Define $[u] + [v] = [u + v]$ and $a[u] = [au]$ for any $a \in \mathbb{R}$ and $u, v \in V$. Prove that V/Z is a vector space.