Week 11

MATH 2040

December 2, 2020

1 Problems

1. Let V be an inner product space over $\mathbb{C}$, show that

   (a) If $x, y \in V$ with $\langle x, v \rangle = \langle y, v \rangle$ for all $v \in V$, then $x = y$.

   (b) If $\{v_1, \cdots, v_n\}$ is an orthogonal basis of V, then $x = \sum_{i=1}^n \frac{\langle x, v_i \rangle}{|v_i|^2} v_i$.

   (c) $\langle x, y \rangle = \sum_{i=i}^n \frac{\langle x, v_i \rangle \langle y, v_i \rangle}{|v_i|^2}$ for orthogonal basis $\{v_i\}$.

   Ans:

   (a) Since $\langle x, v \rangle = \langle y, v \rangle$, $\langle x - y, v \rangle = 0$ holds for all $v \in V$. Let $v = x - y$ then we have $\langle x - y, x - y \rangle = 0$, which means $x - y = 0$ and so $x = y$.

   (b) $\{v_i\}$ is an orthogonal basis, so $x = \sum_{i=1}^n a_i v_i$. On the other hand,

   $$\langle x, v_i \rangle = \sum_{j=1}^n a_j \langle v_j, v_i \rangle = a_i \langle v_i, v_i \rangle,$$

   so $a_i = \frac{\langle x, v_i \rangle}{|v_i|^2}$ and $x = \sum_{i=1}^n \frac{\langle x, v_i \rangle}{|v_i|^2} v_i$.

   (c) from (b) we write $x = \sum_{i=1}^n \frac{\langle x, v_i \rangle}{|v_i|^2} v_i$ and $y = \sum_{i=1}^n \frac{\langle y, v_i \rangle}{|v_i|^2} v_i$, then

   $$\langle x, y \rangle = \sum_{i=i}^n \frac{\langle x, v_i \rangle \langle y, v_i \rangle}{|v_i|^4} \langle v_i, v_i \rangle = \sum_{i=i}^n \frac{\langle x, v_i \rangle \langle y, v_i \rangle}{|v_i|^2}$$

2. For $P_2(\mathbb{R})$ equiped with inner product

$$\langle f, g\rangle = \int_0^2 f(t)g(t)dt,$$

find a standard orthogonal basis.

Ans: We know that $\{1, t, t^2\}$ is a basis of $P_2(\mathbb{R})$ then do gram-schmit process to it to get standard orthogonal basis $\{v_1, v_2, v_3\}$.

$v_1 = \frac{\sqrt{2}}{2}$ since $\int_0^2 \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} dt = 1$.

Then $\langle t, v_1\rangle = \int_0^2 \frac{\sqrt{2}}{2} t dt = \sqrt{2}$ and $\|t - \langle t, v_1\rangle v_1\| = \|t-1\| = \sqrt{\int_0^2 (t-1)^2 dt} = \frac{\sqrt{6}}{3}$, so $v_2 = \frac{t - \langle t, v_1\rangle v_1}{\|t - \langle t, v_1\rangle v_1\|} = \frac{\sqrt{6}}{2}(t-1)$.

Then $\langle t^2, v_1\rangle = \int_0^2 \frac{\sqrt{2}}{2} t^2 dt = \frac{4\sqrt{2}}{3}$, $\langle t^2, v_2\rangle = \int_0^2 \frac{\sqrt{6}}{2}(t^3 - t^2)dt = \frac{2\sqrt{6}}{3}$ and $\|t^2 - \langle t^2, v_1\rangle v_1 - \langle t^2, v_2\rangle v_2\| = \|t^2 - 2t + \frac{2}{3}\| = \sqrt{\int_0^2 (t^2 - 2t + \frac{2}{3})^2 dt} = \frac{2\sqrt{10}}{15}$, so $v_3 = \frac{t^2 - \langle t^2, v_1\rangle v_1 - \langle t^2, v_2\rangle v_2}{\|t^2 - \langle t^2, v_1\rangle v_1 - \langle t^2, v_2\rangle v_2\|} = \frac{3\sqrt{10}}{4}(t^2 - 2t + \frac{2}{3})$.

Therefore, $\{\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}(t-1), \frac{3\sqrt{10}}{4}(t^2 - 2t + \frac{2}{3})\}$ is a standard orthogonal basis.

3. Show that $\langle A, B\rangle = \text{tr}(AB^*)$ defines an inner product space on $M_{m\times n}(\mathbb{C})$.

   Ans: We only need to check the definition of inner product

   (a) $\text{tr}(AB^*)$ is linear

   (b) $\text{tr}(AB^*) = \overline{\text{tr}(BA^*)}$

   (c) $\text{tr}(AA^*) > 0$ when $A \neq 0$

   For (a), we know that $(A + C)B^* = AB^* + CB^*$ so $\text{tr}((A + C)B^*) = \text{tr}(AB^* + CB^*) = \text{tr}(AB^*) + \text{tr}(CB^*)$, then $\langle A + C, B\rangle = \langle A, B\rangle + \langle C, B\rangle$. And since $\text{tr}(cAB^*) = c\text{tr}(AB^*)$ where $c \in C$ so $\langle cA, B\rangle = c\langle A, B\rangle$. Therefore $\text{tr}(AB^*)$ is linear.

   For (b), $\text{tr}(AB^*) = \sum_i (AB^*)_{ii} = \sum_{i,j} A_{ij}B^*_{ji} = \sum_{i,j} A_{ij}\bar{B}_{ij}$, then $\overline{\text{tr}(BA^*)} = \overline{\sum_{ij} B_{ij}\bar{B}_{ij}} = \sum_{i,j} A_{ij}\bar{B}_{ij}$, so we have $\text{tr}(AB^*) = \overline{\text{tr}(B^*A)}$.

   For (c), $\text{tr}(AA^*) = \sum_{ij} A_{ij}\bar{A}_{ij} = \sum_{i,j} \|A\|^2$, so $\text{tr}(AA^*) > 0$ if and only if $A \neq 0$.

   Therefore, this is a well-defined inner product.

4. Let $T : V \to W$ be linear and $T^* : W \to V$ be the adjoint of T, show that

   (a) $R(T^*)^\perp = N(T)$

   (b) $R(T^*) = N(T)^\perp$

   (c) $R(T) = N(T^*)^\perp$

   (d) $R(T)^\perp = N(T^*)$

   Ans: For (a), suppose $v \in R(T^*)^\perp$, by definition we have $\langle v, T*(w)\rangle = 0$, $\forall w \in W$. Then $\langle T(v), w\rangle = 0$ for any W, which means $T(v) = 0$ and $v \in N(T)$.

   Suppose $v \in N(T)$, by definition we have $T(v) = 0$. Then $\langle v, T^*(w)\rangle = \langle T(v), w\rangle = \langle 0, w\rangle = 0$, which means $v \in R(T^*)^\perp$.

   Therefore, (a) is true. Since $(V^\perp)^\perp = V$, we have $R(T^*) = (R(T^*)^\perp)^\perp = N(T)^\perp$, so (b) is true. Since $T^{**} = T$, from (a) we have $R(T)^\perp = R(T^{**})^\perp = N(T^*)$, which means (d) is true. And from (b) we have $R(T^*) = R(T^{***}) = N(T^{**})^\perp = N(T)^\perp$, which means (c) is true.

5. Show that $N(T) = N(T^*T)$.

   Ans:

   $\Rightarrow$: For $v \in N(T)$, $T(v) = 0$, then $T^*(T(v)) = T^*(0) = 0$, which means $v \in N(T^*T)$.

   $\Leftarrow$: For $v \in N(T^*T)$, $T^*T(v) = 0$, then $\|T(v)\|^2 = \langle T(v), T(v)\rangle = \langle T^*T(v), v\rangle = \langle 0, v\rangle = 0$, which means $T(v) = 0$ and $v \in N(T)$.