Week 10

MATH 2040

November 17, 2020

1 Review

1. **Cayley-Hamiton theorem:** $f_A(t) = \det(A - tI) \Rightarrow f_A(A) = 0$.
2. **Inner product space over $\mathbb{R}$:**
   (a) $\langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$
   (b) $\langle u, av + bw\rangle = a\langle u, v\rangle + b\langle u, w\rangle$
   (c) $\langle u, v\rangle = \langle v, u\rangle$
   (d) $\langle u, u\rangle = \|u\|^2 \geq 0$
3. **Inner product space over $\mathbb{C}$:**
   (a) $\langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$
   (b) $\langle u, av + bw\rangle = \bar{a}\langle u, v\rangle + \bar{b}\langle u, w\rangle$
   (c) $\langle u, v\rangle = \overline{\langle v, u\rangle}$
   (d) $\langle u, u\rangle = \|u\|^2 \geq 0$
4. **Cachy-schwartz inequality:** $|\langle u, v\rangle| \leq \|u\|\|v\|$
5. **Triangle inequality:** $\|u + v\| \leq \|u\| + \|v\|$

2 Problems

1. (Application of Cayley-Hamiton theorem) Let $A = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$

   (a) Verify the Cayley Hamiton theorem.

   (b) If t^n is devided by $f_A(t)$, the remainder is $at + b$, find a and b.

   (c) Calculate A^n.

   Ans:

   (a) We have that $f_A(t) = \det(A - tI) = \begin{vmatrix} -1-t & 2 \\ 1 & -t \end{vmatrix} = t^2 + t - 3 = (t-1)(t+2)$, so $f_A(A) = (A - I)(A + 2I) = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

   (b) Suppose $t^n = f_A(t)q(t) + at + b$, where $q(t)$ is a ploynomial, $a, b \in \mathbb{R}$. Let t be 1 and -2, we have that

   $$\begin{aligned} 1 &= a + b \\ (-2)^n &= -2a + b, \end{aligned}$$

   so $a = \frac{1-(-2)^n}{3}$ and $b = \frac{2+(-2)^n}{3}$.

   (c) $A^n = f_A(A)q(A) + aA + bI$, by Cayley-Hamiton theorem $f_A(A) = 0$, then

   $$\begin{aligned} A^n = aA + bI &= \frac{1}{3} \begin{pmatrix} -1 + (-2)^n & 2 + (-2)^{n+1} \\ 1 - (-2)^n & 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 2 + (-2)^n & 0 \\ 0 & 2 + (-2)^n \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 1 - (-2)^{n+1} & 2 + (-2)^{n+1} \\ 1 - (-2)^n & 2 + (-2)^n \end{pmatrix} \end{aligned}$$

2. (Special case) Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$

   (a) If t^n is devided by $f_A(t)$, the remainder is $at + b$, find a and b.

   (b) Calculate A^n.

   Ans:

   (a) We have that $f_A(t) = \begin{vmatrix} 2-t & 1 \\ -1 & -t \end{vmatrix} = t^2 - 2t + 1 = (t-1)^2$, suppose $t^n = f_A(t)q(t) + at + b$, then consider the derivative of $(t^n)' = nt^{n-1} = f'_A(t)q(t) + f_A(t)q'(t) + a$. Let $t = 1$, $f_A(1) = f'_A(1) = 0$, then

   $$\begin{aligned} 1 &= a + b \\ n &= a, \end{aligned}$$

   so $a = n$ and $b = 1 - n$.

   (b) Same with last question, we have that

   $$\begin{aligned} A^n = aA + bI &= \begin{pmatrix} 2n & n \\ -n & 0 \end{pmatrix} + \begin{pmatrix} 1-n & 0 \\ 0 & 1-n \end{pmatrix} \\ &= \begin{pmatrix} 1+n & n \\ -n & 1-n \end{pmatrix} \end{aligned}$$

3. Let $A \in M_{2\times 2}(\mathbb{F})$, suppose $\det(A) \neq 0$, show that $A^{-1} = \frac{1}{\det(A)}(\mathrm{tr}(A)I - A)$.

   Ans: Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $tr(A) = a + d$, $\det(A) = ad - bc$ and $f_A(t) = \det(A - tI) = t^2 - (a+d)t + ad - bc$. By Cayley-Hamiton theorem, $f_A(A) = A^2 - \mathrm{tr}(A)A + \det(A)I = 0$. Since $\det(A) \neq 0$, A^{-1} exists, $A - tr(A) + \det(A)A^{-1} = 0$, so $A^{-1} = \frac{1}{\det(A)}(\mathrm{tr}(A)I - A)$.

4. (a) $\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$, $x, y \in \mathbb{R}$.

(b) $\langle x, y\rangle = \frac{1}{4}\sum_{k=1}^{4} i^k \|x + i^k y\|^2$, $x, y \in \mathbb{C}$.

Ans:

(a) When $x, y \in \mathbb{R}$, we know that

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle = \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle \\ \|x-y\|^2 &= \langle x-y, x-y\rangle = \langle x, x\rangle - 2\langle x, y\rangle + \langle y, y\rangle,\end{aligned}$$

so $\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) = \langle x, y\rangle$.

(b) When $x, y \in \mathbb{C}$, we know that

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ i\|x+iy\|^2 &= i\langle x+iy, x+iy\rangle = i\langle x, x\rangle + \langle x, y\rangle - \langle y, x\rangle + i\langle y, y\rangle \\ -\|x-y\|^2 &= -\langle x-y, x-y\rangle = -\langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle - \langle y, y\rangle \\ -i\|x-iy\|^2 &= -i\langle x-iy, x-iy\rangle = -i\langle x, x\rangle + \langle x, y\rangle - \langle y, x\rangle - i\langle y, y\rangle\end{aligned}$$

so $\frac{1}{4}\sum_{k=1}^{4} i^k \|x + i^k y\|^2 = \langle x, y\rangle$.

5. Given inner product spaces V and W over $\mathbb{C}$, suppose $f : V \to W$ with $\langle f(u), f(v)\rangle = \langle u, v\rangle$ for any $u, v \in V$.

   (a) Show that f is linear.

   (b) Show that f is injective.

   Ans:

   (a) i. $\|f(0)\|^2 = \langle f(0), f(0)\rangle = \langle 0, 0\rangle = \|0\|^2 = 0$, so $f(0) = 0$

   ii.

$$\begin{aligned}\|f(u+v) - f(u) - f(v)\|^2 =&\langle f(u+v), f(u+v)\rangle + \langle f(u), f(u)\rangle + \langle f(v), f(v)\rangle \\ &- \langle f(u+v), f(u)\rangle - \langle f(u+v), f(v)\rangle - \langle f(u), f(u+v)\rangle \\ &+ \langle f(u), f(v)\rangle - \langle f(v), f(u+v)\rangle + \langle f(v), f(u)\rangle \\ =&\langle u+v, u+v\rangle + \langle u, u\rangle + \langle v, v\rangle \\ &- \langle u+v, u\rangle - \langle u+v, v\rangle - \langle u, u+v\rangle \\ &+ \langle u, v\rangle - \langle v, u+v\rangle + \langle v, u\rangle \\ =&\langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle + \langle u, u\rangle + \langle v, v\rangle \\ &- \langle u, u\rangle - \langle v, u\rangle - \langle u, v\rangle - \langle v, v\rangle - \langle u, u\rangle - \langle u, v\rangle \\ &+ \langle u, v\rangle - \langle v, u\rangle - \langle v, v\rangle + \langle v, u\rangle \\ =&0,\end{aligned}$$

   so $f(u+v) = f(u) + f(v)$.

   iii.

$$\begin{aligned}\|af(u) - f(au)\|^2 =&\langle af(u), af(u)\rangle + \langle f(au), f(au)\rangle - \langle af(u), f(au)\rangle - \langle f(au), af(u)\rangle \\ =&|a|^2\langle f(u), f(u)\rangle + \langle f(au), f(au)\rangle - a\langle f(u), f(au)\rangle - \bar{a}\langle f(au), f(u)\rangle \\ =&|a|^2\langle u, u\rangle + \langle au, au\rangle - a\langle u, au\rangle - \bar{a}\langle au, u\rangle \\ =&|a|^2\langle u, u\rangle + |a|^2\langle u, u\rangle - |a|^2\langle u, u\rangle - |a|^2\langle u, u\rangle \\ =&0,\end{aligned}$$

   so $f(au) = af(u)$.

   Thereforce, f is linear.

   (b) $\forall w \in N(f)$, $f(w) = 0$, then $\|f(w)\|^2 = \langle f(w), f(w)\rangle = \langle w, w\rangle = \|w\| = 0$, so $w = 0$ and $N(f) = \{0\}$, f is injective.