THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2040A/B (First Term, 2020-2021)
Linear Algebra II
Solution to Homework 11

Sec. 6.5

2 Q: For each of the following matrices A, find an orthogonal or unitary matrix P and a diagonal matrix D such that $P^*AP = D$.

(c)

$$\begin{pmatrix} 2 & 3-3i \\ 3+3i & 5 \end{pmatrix}$$

(d)

$$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

(e)

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Sol: (c) The characteristic polynomial of A is

$$(2-t)(5-t) - (3-3i)(3+3i) = t^2 - 7t - 8 = (t-8)(t+1).$$

Hence, $-1, 8$ are all the eigenvalues of A. Note that for any scalars a, b,

$$3\begin{pmatrix} -2 & 1-i \\ 1+i & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -6 & 3-3i \\ 3+3i & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (A-8I)\begin{pmatrix} a \\ b \end{pmatrix} = \vec{0}$$

if and only if $b = (1+i)a$. In particular, $u = (1, 1+i)$ is an eigenvector of A corresponding to eigenvalue 8.

$$\|u\| = \sqrt{1\bar{1} + (1+i)\overline{(1+i)}} = \sqrt{3}.$$

On the other hand, for any scalars a, b,

$$3\begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 & 3-3i \\ 3+3i & 6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (A+I)\begin{pmatrix} a \\ b \end{pmatrix} = \vec{0}$$

if and only if $a = (i-1)b$. In particular, $v = (i-1, 1)$ is an eigenvector of A corresponding to eigenvalue -1.

$$\|v\| = \sqrt{(i-1)\overline{(i-1)} + 1\bar{1}} = \sqrt{3}.$$

Then

$$P = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & i-1 \\ i+1 & 1 \end{pmatrix}$$

is a unitary matrix and

$$D = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}$$

is a diagonal matrix such that $P^* A P = D$.

(d) The characteristic polynomial of A is

$$\det \begin{pmatrix} -t & 2 & 2 \\ 2 & -t & 2 \\ 2 & 2 & -t \end{pmatrix} = (4-t)(2+t)^2.$$

Hence, $4, 1$ are all the eigenvalues of A. It is clear that $u = (1,1,1)$ is an eigenvector of A corresponding to eigenvalue 4.

$$\|u\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}.$$

Note that for any scalars a, b, c,

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (A + 2I) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \vec{0}$$

if and only if $a + b + c = 0$. Then we see that $v = (1, -1, 0)$ is an eigenvector of A corresponding to eigenvalue 1. We would like to find a further eigenvector $w = (a', b', c')$ of A corresponding to 1 such that $\langle v, w \rangle = 0$, i.e. $a' - b' = 0$. Then we see that $w = (1, 1, -2)$ is such a eigenvector.

$$\|v\| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}.$$
$$\|w\| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}.$$

Then

$$P = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 1 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 1 & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$$

is a unitary matrix and

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

is a diagonal matrix such that $P^* A P = D$.

(e) The characteristic polynomial of A is

$$\det\begin{pmatrix} 2-t & 1 & 1 \\ 1 & 2-t & 1 \\ 1 & 1 & 2-t \end{pmatrix} = \det\begin{pmatrix} 2-t & 1 & 1 \\ t-1 & 1-t & 0 \\ t-1 & 0 & 1-t \end{pmatrix} = \det\begin{pmatrix} 4-t & 1 & 1 \\ 0 & 1-t & 0 \\ 0 & 0 & 1-t \end{pmatrix}$$
$$= (4-t)(1-t)^2.$$

Hence, $4, 1$ are all the eigenvalues of A. It is clear that $u = (1, 1, 1)$ is an eigenvector of A corresponding to eigenvalue 4.

$$\|u\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}.$$

Note that for any scalars a, b, c,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (A - I)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \vec{0}$$

if and only if $a + b + c = 0$. Then we see that $v = (1, -1, 0)$ is an eigenvector of A corresponding to eigenvalue 1. We would like to find a further eigenvector $w = (a', b', c')$ of A corresponding to 1 such that $\langle v, w\rangle = 0$, i.e. $a' - b' = 0$. Then we see that $w = (1, 1, -2)$ is such a eigenvector.

$$\|v\| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}.$$
$$\|w\| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}.$$

Then

$$P = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 1 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 1 & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$$

is a unitary matrix and

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a diagonal matrix such that $P^*AP = D$.

6 Q: Let V be the inner product space of complex-valued continuous functions on [0,1] with the inner product

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}dt$$

Let $h \in$ V, and define T : V $\to$ V by T(f) = hf. Prove that T is a unitary operator if and only if $|h(t)| = 1$ for $0 \le t \le 1$.

Sol: If T is unitary, we must have

$$0 = \|T(f)\|^2 - \|f\|^2 = \int_0^1 |h|^2|f|^2dt - \int_0^1 |f|^2dt$$
$$= \int_0^1 \left(1 - |h|^2\right)|f|^2dt$$

for all $f \in V$. Pick $f = \left(1 - |h|^2\right)^{\frac{1}{2}}$ and get $1 - |h|^2 = 0$ and so $|h| = 1$. Conversely, if $|h| = 1$, we have

$$\begin{aligned} \|T(f)\|^2 - \|f\|^2 &= \int_0^1 |h|^2 |f|^2 dt - \int_0^1 |f|^2 dt \\ &= \int_0^1 \left(1 - |h|^2\right) |f|^2 dt = 0 \end{aligned}$$

and so T is unitary.

7 Q: Prove that if T is a unitary operator on a finite-dimensional inner product space V, then T has a unitary square root; that is, there exists a unitary operator U such that $\mathrm{T} = \mathrm{U}^2$.

Sol: By the Corollary 2 after Theorem 6.18 , we may find an orthonormal basis β such that

$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}.$$

Also, since the eigenvalue λ_i has its absolute value 1, we may find some number μ_i such that $\mu_i^2 = \lambda_i$ and $|\mu_i| = 1$. Denote

$$D = \begin{pmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mu_n \end{pmatrix}$$

to be an unitary operator. Now pick U to be the matrix whose matrix representation with respect to β is D. Thus U is unitary and $U^2 = T$.

12 Q: Let A be an $n \times n$ real symmetric or complex normal matrix. Prove that

$$\det(A) = \prod_{i=1}^{n} \lambda_i$$

where the λ_i 's are the (not necessarily distinct) eigenvalues of A.

Sol: By Theorem 6.19 and Theorem 6.20 we know that A may be diagonalized as $P^*AP = D$. Here D is a diagonal matrix whose diagonal entries consist of all eigenvalues. Now we have

$$\det(A) = \det\left(PDP^*\right) = \det(D) = \prod_{i=1}^{n} \lambda_i$$

13 Q: Suppose that A and B are diagonalizable matrices. Prove or disprove that A is similar to B if and only if A and B are unitarily equivalent.

Sol: The necessity is false. For example, the two matrices $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ and

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

are similar. But they are not unitary since one is symmetric but the other is not.

Sec. 6.6

2 Q: Let V = R^2, W = span({(1, 2)}), and β be the standard ordered basis for V. Compute $[\mathrm{T}]_\beta$, where T is the orthogonal projection of V on W. Do the same for V = R^3 and W = span({(1, 0, 1)}).

Sol: We could calculate the projection of (1, 0) and (0, 1):

$$\frac{\langle (1,0),(1,2)\rangle}{\|(1,2)\|^2}(1,2) = \frac{1}{5}(1,2)$$

and

$$\frac{\langle (0,1),(1,2)\rangle}{\|(1,2)\|^2}(1,2) = \frac{2}{5}(1,2)$$

respectively by Theorem 6.6, So we have

$$[T]_\beta = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

On the other hand, we may do the same on (1, 0, 0), (0, 1, 0), and (1,0,0) with respect to the new subspace $W = \mathrm{span}(\{(1,0,1)\})$. First we compute

$$\frac{\langle (1,0,0),(1,0,1)\rangle}{\|(1,0,1)\|^2}(1,0,1) = \frac{1}{2}(1,0,1)$$
$$\frac{\langle (0,1,0),(1,0,1)\rangle}{\|(1,0,1)\|^2}(1,0,1) = 0(1,0,1)$$

and

$$\frac{\langle (0,0,1),(1,0,1)\rangle}{\|(1,0,1)\|^2}(1,0,1) = \frac{1}{2}(1,0,1).$$

Hence the matrix would be

$$[T]_\beta = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

4 Q: Let W be a finite-dimensional subspace of an inner product space V. Show that if T is the orthogonal projection of V on W, then $I - T$ is the orthogonal projection of V on $W^\perp$.

Sol: Fix $v \in V$. Then $\exists$ unique $w \in W$ and unique $u \in W^\perp$ such that $v = w + u$. As T is the orthogonal projection of V on W, $w = T(v)$ and thus $u = v - w = (I - T)(v)$. Therefore, $I - T$ is a projection of V on $W^\perp$ along $W = (W^\perp)^\perp$, which implies that $I - T$ is the orthogonal projection of V on $W^\perp$.

6 Q: Let T be a normal operator on a finite-dimensional inner product space. Prove that if T is a projection, then T is also an orthogonal projection.

Sol: Let V be the domain of the operator T. Fix $u \in \mathsf{N}(T)$ and $w \in \mathsf{R}(T)$. We claim that $\langle u, v\rangle = 0$. If either u or w is the zero vector, then we are done. Now suppose $u \neq \vec{0}$ and $w \neq \vec{0}$. As $T(u) = \vec{0}$ and $T(w) = w$, u is indeed an eigenvector of T corresponding to the eigenvalue 0, while w is an eigenvector of T corresponding to the eigenvalue 1. By Theorem 6.15, $\langle u, w\rangle = 0$. Therefore, $\mathsf{N}(T)$ and $\mathsf{R}(T)$ are orthogonal, whence T is an orthogonal projection.

7 Q: Let T be a normal operator on a finite-dimensional complex inner product space V. Use the spectral decomposition $\lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k$ of T to prove the following results.

(a) If g is a polynomial, then

$$g(T) = \sum_{i=1}^{k} g(\lambda_i) T_i.$$

(b) If $T^n = T_0$ for some n, then $T = T_0$.

(c) Let U be a linear operator on V. Then U commutes with T if and only if U commutes with each T_i.

(d) There exists a normal operator U on V such that $U^2 = T$.

(e) T is invertible if and only if $\lambda_i \neq 0$ for $1 \leq i \leq k$.

(f) T is a projection if and only if every eigenvalue of T is 1 or 0.

(g) $T = -T^*$ if and only if every λ_i is an imaginary number.

Sol: (a) Note that $T^0 = I = \sum_{i=1}^{k} T_i$. $\forall j \in \mathbb{Z}^+$,

$$T^j = \sum_{i_1=1}^{k} \cdots \sum_{i_j=1}^{k} \lambda_{i_1} \cdots \lambda_{i_j} T_{i_1} \cdots T_{i_j} = \sum_{i_1=1}^{k} \cdots \sum_{i_j=1}^{k} \lambda_{i_1} \cdots \lambda_{i_j} \delta_{i_1 i_2} \delta_{i_1 i_3} \cdots \delta_{i_1 i_j} T_{i_1}$$
$$= \sum_{i=1}^{k} \lambda_i^j T_i.$$

Write $g(t) = a_n t^n + \cdots + a_1 t + a_0$, where $a_0, ..., a_n \in \mathbb{C}$. Then

$$g(T) = a_n T^n + \cdots + a_1 T + a_0 I = a_n \sum_{i=1}^{k} \lambda_i^n T_i + \cdots a_1 \sum_{i=1}^{k} \lambda_i T_i + a_0 \sum_{i=1}^{k} T_i$$
$$= \sum_{i=1}^{k} (a_n \lambda_i^n + \cdots a_1 \lambda_i + a_0) T_i = \sum_{i=1}^{k} g(\lambda_i) T_i.$$

(b) Suppose $T^n = T_0$ for some n. Then $\sum_{i=1}^{k} \lambda_i^n T_i = T_0$. It implies that $\lambda_1^n = \cdots = \lambda_k^n = 0$, whence $\lambda_1 = \cdots = \lambda_k = 0$. Therefore, $T = T_0$.

(c) ($\Rightarrow$) Since T, U commute, a T-invariant subspace of V is also U-invariant. Fix $v \in V$. $\forall i \in \{1, ..., k\}$, we have

$$T_i U(v) + (T - (\lambda_i - 1) T_i) U(v) = TU(v) = UT(v) = UT_i(v) + U(T - (\lambda_i - 1) T_i)(v)$$

and therefore $T_i U(v) = UT_i(v)$.

($\Leftarrow$) We have

$$UT = \lambda_1 UT_1 + \cdots + \lambda_k UT_k = \lambda_1 T_1 U + \cdots + \lambda_k T_k U = TU.$$

(d) $\forall i \in \{1, ..., k\}$, choose $\mu_i \in \mathbb{C}$ such that $\mu_i^2 = \lambda_i$. Define $U = \mu_1 T_1 + \cdots + \mu_k T_k$. By Gram-Schmidt Orthogonalization Process and Theorem 6.16, U is normal. Using the result of (a), $U^2 = \mu_1^2 T_1 + \cdots + \mu_k^2 T_k = \lambda_1 T_1 + \cdots + \lambda_k T_k = T$.

(e) ($\Rightarrow$) In particular, $\mathsf{N}(T) = \{\vec{0}\}$. Then 0 is not an eigenvalue of T, whence $\lambda_i \neq 0$ for $1 \leq i \leq k$.

($\Leftarrow$) It means that 0 is not an eigenvalue of T. So if $v \in \mathsf{N}(T)$, then $T(v) = \vec{0} = 0 \cdot v$, forcing that $v = \vec{0}$. T is then one-to-one. As V is finite-dimensional, T is also onto. Then T is invertible.

(f) ($\Rightarrow$) Suppose $\lambda \in \mathbb{C}$ is an eigenvalue of T. Then $\exists v \in V$ such that $v \neq \vec{0}$ and $T(v) = \lambda v$. As T is a projection, $\lambda v = T(v) = T^2(v) = \lambda^2 v$, whence $\lambda(\lambda - 1)v = \vec{0}$. As $v \neq \vec{0}$, $\lambda(\lambda - 1) = 0$, whence either $\lambda = 1$ or $\lambda = 0$.

($\Leftarrow$) Case (1): Suppose 1 is an eigenvalue of T. Then without loss of generality we can assume $\lambda_1 = 1$ and $\lambda_i = 0$ for any $1 < i \leq k$. Then $T = T_1$ is a projection.

Case (2): Suppose 1 is not eigenvalue of T. Then without loss of generality we can assume $\lambda_i = 0$ for any $1 \leq i \leq k$ and hence T is the zero transformation, which is a projection as well.

(g) ($\Rightarrow$) Fix $i \in \{1, ..., k\}$. Fix v_i with $v_i \neq \vec{0}$ and $T(v_i) = \lambda_i v_i$. Then $T^*(v_i) = \overline{\lambda_i} v_i$. We have $\lambda_i v_i = T(v_i) = -T^*(v_i) = -\overline{\lambda}_i v_i$. But $v_i \neq \vec{0}$. Thus, $\lambda_i = -\overline{\lambda}_i$. It means that λ_i is an imaginary number.

($\Leftarrow$) Fix $v \in V$. Then $\exists v_1, ..., v_k \in V$ such that $T(v_i) = \lambda_i v_i$ $\forall i \in \{1, ..., k\}$ and $v = v_1 + \cdots + v_k$. We have

$$-T^*(v) = -T^*(v_1) - \cdots - T^*(v_k) = -\overline{\lambda}_1 v_1 - \cdots - \overline{\lambda_k} v_k = \lambda_1 v_1 + \cdots + \lambda_k v_k = T(v).$$

Therefore, $T = -T^*$.

10 Q: *Simultaneous diagonalization.* Let U and T be normal operators on a finite-dimensional complex inner product space V such that $TU = UT$. Prove that there exists an orthonormal basis for V consisting of vectors that are eigenvectors of both T and U.

Sol: Let $\lambda_1, ..., \lambda_k$ be all the distinct eigenvalues of T. $\forall i \in \{1, ..., k\}$, let E_{λ_i} be the eigenspace of T corresponding to the eigenvalue λ_i. By Theorem 6.16, we have an orthogonal decomposition

$$V = \mathsf{E}_{\lambda_1} \oplus \cdots \oplus \mathsf{E}_{\lambda_k}.$$

Fix $i \in \{1, ..., k\}$. Since $TU = UT$, E_{λ_i} is U-invariant. Note that E_{λ_i} is the eigenspace of T^* corresponding to eigenvalue $\overline{\lambda}_i$. We also have $T^*U^* = (UT)^* = (TU)^* = U^*T^*$ and thus E_{λ_i} is also U^*-invariant. Then by Exercise 7 in Sec. 6.4, $U_{\mathsf{E}_{\lambda_i}}$ is normal because U is normal. By Theorem 6.16, $\exists$ orthonormal basis $\{v_{ii}, ..., v_{in_i}\}$ of $U_{\mathsf{E}_{\lambda_i}}$ for E_{λ_i} such that $v_{ii}, ..., v_{in_i}$ are eigenvectors of $U_{\mathsf{E}_{\lambda_i}}$. Then

$$\beta = \{v_{11}, ..., v_{1n_1}, ..., v_{k1}, ..., v_{kn_k}\}$$

is an orthonormal basis for V such that $\forall i \in \{1, ..., k\}$, $\forall j \in \{1, ..., n_i\}$, v_{ij} is an eigenvector of both U and T.