Part I. Basic knowledges.

1. (**10pts**) Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. Define addition of elements of $V$ coordinatewise, and for $(a_1, a_2)$ in $V$ and $c \in \mathbb{R}$, define

$$c(a_1, a_2) = \begin{cases} (0,0) & \text{if } c = 0 \\ \left(ca_1, \frac{a_2}{c}\right) & \text{if } c \neq 0. \end{cases}$$

   Is $V$ a vector space over $\mathbb{R}$ with these operations? Justify your answer.

2. (**10pts**) Is the set

$$W = \{f(x) \in P(F) : f(x) = 0 \text{ or } f(x) \text{ has degree } n\}$$

   a subspace of $P(F)$ if $n \geq 1$? Justify your answer.

3. (**10pts**) Let $S = \{2 + 3x + 5x^2 + 7x^3, 1 + 2x + 3x^2 + 3x^3, 5 + 8x + 13x^2 + 17x^3, 1 + 2x + 1x^2 + 1x^3\}$ in $V = P_5(\mathbb{R})$. Show that $S$ is linear dependent. Reduce it to a linear independent set $\beta$ such that $\text{Span}(\beta) = \text{Span}(S)$. Finally, extend $\beta$ to a basis for $V$. Please explain your answers with details and show all the steps including your calculation.

4. (**10pts**) Let $U$ be the subspace of $\mathbb{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

   Find a basis of $U$.

5. (**10pts**) Let $V = M_{2\times 2}(F)$. Define

$$W_1 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in V : a, b, c \in F \right\}$$

   and

$$W_2 = \left\{ \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \in V : a, b \in F \right\}.$$

   Prove that $W_1$ and $W_2$ are subspaces of $V$, and find the dimensions of $W_1$, $W_2$, $W_1 + W_2$, and $W_1 \cap W_2$.

Part II. Theory knowledge.

1. (**10pts**) Suppose that $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is linearly independent in a vector space $V$ and $\mathbf{w} \in V$. Prove that if
$$\{\mathbf{v}_1 + \mathbf{w}, ..., \mathbf{v}_n + \mathbf{w}\}$$
is linearly dependent, then $\mathbf{w} \in \mathrm{span}(\{\mathbf{v}_1, ..., \mathbf{v}_n\})$.

2. (**10pts**) Let $V$ be a vector space over $\mathbb{R}$. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\} \subset V$ be a basis for $V$. Suppose for each $1 \leq i \leq m$,
$$\mathbf{w}_i = a_{i1}\mathbf{v}_1 + a_{i2}\mathbf{v}_2 + ... + a_{in}\mathbf{v}_n$$
where $a_{ij} \in \mathbb{R}$. Let $A \in M_{m\times n}$ whose $i$-th row $j$-th entry is given by $a_{ij}$. Prove that $\{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_m\}$ is a basis for $V$ if and only if $m = n$ and $A$ is non-singular.

3. (**10pts**) Let $V$ be a vector space over $F$. Suppose that $W_1$, $W_2$ and $W_3$ are the subspaces of $V$. Prove that $W_1 \cup W_2 \cup W_3$ is a subspace of $V$ if and only if
$$\text{either } W_1 \cup W_2 \subset W_3, \text{ or } W_2 \cup W_3 \subset W_1, \text{ or } W_3 \cup W_1 \subset W_2.$$

4. (**10pts**) Prove that a vector space $V$ is infinite dimensional if and only if there is a sequence $\mathbf{v}_1, \mathbf{v}_2, ...$ of vectors in $V$ such that $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is linearly independent for every positive integer $n$.

5. (**10pts**) Suppose that $f_0, f_1, ..., f_m$ are polynomials in $P_m(F)$ such that $f_j(2) = 0$ for each $j = 0, 1, \cdots, m$. Prove that $\{f_0, f_1, ..., f_m\}$ is a linearly dependent subset of $P_m(F)$.