Lecture 8

$T: V \to W$

Recall:

$$N(T) = \{\vec{x} \in V : T(\vec{x}) = \vec{0}_W\} \subseteq V$$

$$R(T) = \{T(\vec{x}) : \vec{x} \in V\} \subseteq W$$

subspaces

$$\text{Rank}(T) = \dim(R(T))$$

$$\text{Nullity}(T) = \dim(N(T))$$

Rank-Nullity Thm

$$\text{Rank}(T) + \text{Nullity}(T) = \dim(V)$$

$\cancel{\dim(W)}$

Example: Consider $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by:

$$T(f(x)) \overset{\text{def}}{:=} 2f'(x) + \int_0^x 3f(t)\,dt$$

We have

$$R(T) = \operatorname{span}\{T(1), T(x), T(x^2)\}$$

$$= \operatorname{span}\{\underbrace{3x,\ 2+\tfrac{3}{2}x^2,\ 4x+x^3}_{\text{Linear independent}}\}$$

$$\dim(R(T)) = \operatorname{rank}(T) = 3$$

$$\cancelto{3}{\operatorname{Rank}(T)} + \operatorname{Nullity}(T) = \dim(\cancelto{3}{P_2(\mathbb{R})})$$

$$\Rightarrow \operatorname{Nullity}(T) = 0 \Rightarrow N(T) = \{\vec{0}\}$$

$$\Rightarrow T \text{ is one-to-one.}$$

Thm: Let V and W be vector spaces of equal finite-dimensions. Let $T: V \to W$ be a linear transformation.
Then, the following are equivalent:

(a) T is one-to-one

(b) T is onto

(c) $\text{Rank}(T) = \dim(V)$

$\dim(R(T)) \leq \dim(W)$

Proof: T is one-to-one

$\iff \text{Nullity}(T) = 0$ (by previous proposition)

$\iff \text{Rank}(T) + \overset{0}{\cancel{\text{Nullity}(T)}} = \dim(V)$

$\iff \underset{\dim(R(T))}{\underset{\parallel}{\text{Rank}(T)}} = \dim(W) \iff R(T) = W$

$\iff T$ is onto

Example: Show that $\forall\, q(x) \in P(\mathbb{R})$, $\exists\, p(x) \in P(\mathbb{R})$ such that

$\forall$: for all $\qquad$ $\exists$: there exists

$$[(x^2+5x+7)\,p(x)]'' = q(x).$$

$\Updownarrow$

(T is onto)

Consider $T : P(\mathbb{R}) \to P(\mathbb{R})$ defined by:

$$T(p(x)) = [(x^2+5x+7)\,p(x)]''$$

(Exercise: T is linear)

(~~Need to check $N(T) = \{\vec{0}\}$ or Nullity$(T) = 0$~~)

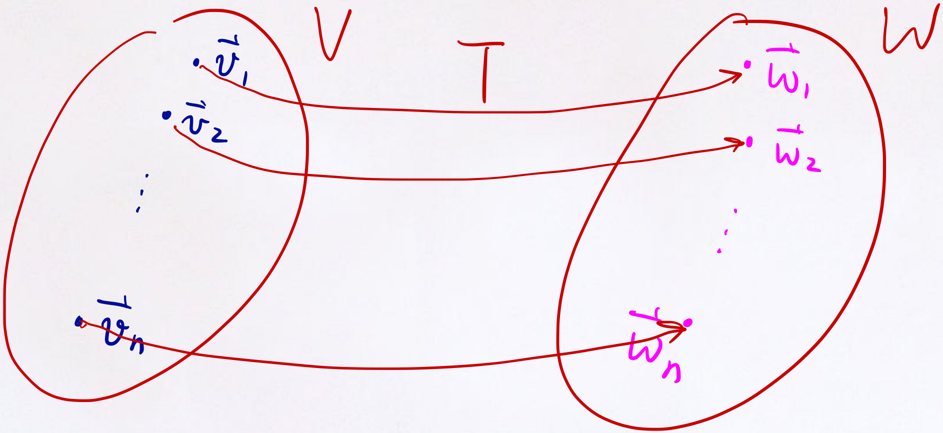
because $\dim(P(\mathbb{R})) = \infty$

Idea: Restrict T to $P_n(\mathbb{R})$: Define: $T : P_n(\mathbb{R}) \to P_n(\mathbb{R})$

such that $T(p(x)) = [(x^2+5x+7)\,p(x)]''$

Remain to show Nullity$(T) = 0$. (Exercise)

Thm: Let V and W be vector spaces. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis of V. Then, given any $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n \in W$, $\exists$ a unique linear transformation $T: V \to W$ such that $T(\vec{v}_i) = \vec{w}_i$ for $i = 1, 2, \ldots, n$



Proof: For $\vec{x} \in V$, $\exists!\ a_1, a_2, \ldots, a_n \in F$ s.t. $\vec{x} = \sum_{i=1}^{n} a_i \vec{v}_i$.

We define $T: V \to W$ by: $T(\vec{x}) = \sum_{i=1}^{n} a_i \vec{w}_i \in W$

- T is linear: For $\vec{x} = \sum_{i=1}^{n} a_i \vec{v}_i \in V$, $\vec{y} = \sum_{i=1}^{n} b_i \vec{v}_i \in V$

and $c \in F$,

we have:

$$
\begin{aligned}
T(c\vec{x} + \vec{y}) &= T\left(\sum_{i=1}^{n} (c a_i + b_i)\vec{v}_i\right) \\
&= \sum_{i=1}^{n} (c a_i + b_i)\vec{w}_i \\
&= c\underbrace{\left(\sum_{i=1}^{n} a_i \vec{w}_i\right)}_{=\,T(\vec{x})} + \underbrace{\left(\sum_{i=1}^{n} b_i \vec{w}_i\right)}_{=\,T(\vec{y})}
\end{aligned}
$$

- By definition, $T(\vec{v}_i) = \vec{w}_i$ for $i = 1, 2, \ldots, n$
- T is unique: Suppose $U : V \to W$ is linear s.t.

$$U(\vec{v}_i) = \vec{w}_i \quad \text{for } \forall i.$$

For any $\vec{x} = \sum_{i=1}^{n} a_i \vec{v}_i \in V$, we have:

$$U(\vec{x}) = \sum_{i=1}^{n} a_i \underbrace{U(\vec{v}_i)}_{=\vec{w}_i} = \sum_{i=1}^{n} a_i \vec{w}_i = T(\vec{x}).$$

$$\therefore \quad U = T.$$

Corollary: Let V be a vector space with a finite basis $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$.

Then any linear transformation from V to another vector space W is completely determined by its values on β.

(That is, if U and T are linear transformations from V to W s.t. $U(\vec{v}_i) = T(\vec{v}_i)$, then $U = T$)