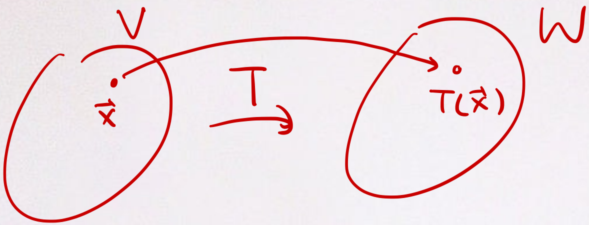
Lecture 7:

Recall: • Linear Transformation $T: V \to W$:

① $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$

② $T(a\vec{x}) = aT(\vec{x})$



• $N(T) \stackrel{\text{def}}{=} \{ \vec{x} \in V : T(\vec{x}) = 0 \} \subset V$

$R(T) \stackrel{\text{def}}{=} \{ T(\vec{x}) : \vec{x} \in V \} \subset W$

Proved that: $N(T)$ and $R(T)$ are subspaces.

Remark: $T: V \to W$ is onto iff $R(T) = W$

(follows from the def)

Proposition: A linear transformation $T: V \to W$ is one-to-one iff $N(T) = \{\vec{0}\}$.

Pf: (Recap: One-to-one $\Leftrightarrow$ " $T(\vec{x}) = T(\vec{y}) \Rightarrow \vec{x} = \vec{y}$ "

($\Rightarrow$) If T is one-to-one, then: for any $\vec{x} \in N(T)$, we have $T(\vec{x}) = \vec{0}_W = T(\vec{0}_V)$

$$\Rightarrow \quad \vec{x} = \vec{0}_V$$

This implies $N(T) = \{\vec{0}_V\}$.

($\Leftarrow$) Suppose $N(T) = \{\vec{0}_V\}$

Let $\vec{x}, \vec{y} \in V$ such that $T(\vec{x}) = T(\vec{y})$.

Then: $T(\vec{x}) - T(\vec{y}) = T(\vec{x} - \vec{y}) = \vec{0}$

This implies $\vec{x} - \vec{y} \in N(T) = \{\vec{0}_V\}$

$\therefore \vec{x} - \vec{y} = \vec{0}_V$ or $\vec{x} = \vec{y}$.

$\therefore$ T is 1-1.

<u>Definition</u>: Let $T: V \to W$ be a linear transformation.

If $N(T)$ and $R(T)$ are finite-dimensional, we define:

- Nullity is denoted as Nullity (T) is the dimension of $N(T)$.
- Rank is denoted as Rank (T) is the dimension of $R(T)$.

<u>Lemma</u>: Let $T: V \to W$ be a linear transformation. If $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for V, then:

$$R(T) = \text{Span}\,(T(\beta)) \overset{\text{def}}{=} \text{span}\,\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)\}$$

Proof: $\because$ $T(\vec{v}_j) \in R(T)$ for $j = 1, 2, \ldots, n$

and $R(T)$ is subspace.

$$\therefore \quad \mathrm{Span}\{\underset{\substack{\uparrow \\ R(T)}}{T(\vec{v}_1)}, \underset{\substack{\uparrow \\ R(T)}}{T(\vec{v}_2)}, \ldots, \underset{\substack{\uparrow \\ R(T)}}{T(\vec{v}_n)}\} \subset R(T)$$

Conversely, let $\underset{\substack{\| \\ T(\vec{x})}}{\vec{w}} \in R(T)$ where $\vec{x} \in V$.

Then: $\exists\ a_1, a_2, \ldots, a_n \in F$ s.t. $\vec{x} = \sum_{j=1}^{n} a_j \vec{v}_j$.

$$\text{So, } \vec{w} = T(\vec{x}) = T\left(\sum_{j=1}^{n} a_j \vec{v}_j\right) = \sum_{j=1}^{n} a_j T(\vec{v}_j) \in \mathrm{Span}\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$$

$$\therefore \quad R(T) \subset \mathrm{Span}(T(\beta)) \qquad \therefore \quad R(T) = \mathrm{Span}(T(\beta))$$

Example: $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by:

$$T(f) = \begin{pmatrix} f(0) & f(1) \\ 0 & f'(0) \end{pmatrix}$$

We have: $R(T) = \text{span}\{T(1), T(x), T(x^2)\}$

$$= \text{span}\underbrace{\left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}}_{\text{Lin. indep.}}$$

$\Rightarrow \text{Rank}(T) = 3$

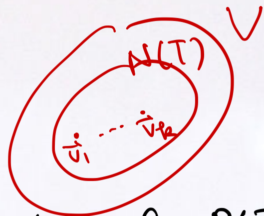
Theorem: (Rank-Nullity Theorem)

Let V and W be vector spaces s.t. V is finite-dimensional.
Then for any linear transformation $T: V \to W$, we have:

$$\text{nullity}(T) + \text{Rank}(T) = \dim(V)$$

Proof: Let $n = \dim(V)$ and $k = \dim(N(T)) \leq n$

Choose a basis $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ for $N(T)$ and extend it to a basis $\{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ for V.



Claim: $S = \{T(\vec{v}_{k+1}), T(\vec{v}_{k+2}), \dots, T(\vec{v}_n)\}$ is a basis for $R(T)$.

① $$R(T) = \mathrm{span}\{\underbrace{T(\vec{v}_1)}_{= 0}, T(\vec{v}_2), \underbrace{T(\vec{v}_k)}_{= 0}, \ldots, T(\vec{v}_n)\}$$

$$= \mathrm{span}\{\underbrace{T(\vec{v}_{k+1}), \ldots, T(\vec{v}_n)}_{S}\} = \mathrm{span}(S)$$

② Now suppose $\exists\, b_{k+1}, b_{k+2}, \ldots, b_n \in F$ s.t.

$$\sum_{i=k+1}^{n} b_i T(\vec{v}_i) = \vec{0}.$$

Then, by linearity, we have: $$T\left(\sum_{i=k+1}^{n} b_i \vec{v}_i\right) = \vec{0}$$

$$\Rightarrow \sum_{i=k+1}^{n} b_i \vec{v}_i \in N(T)$$

$\therefore \displaystyle\sum_{i=k+1}^{n} b_i \vec{v}_i = \sum_{i=1}^{k} c_i \vec{v}_i$ for some $c_1, \dots, c_k \in F$.

But then: $\displaystyle\sum_{i=1}^{k} (-c_i)\, \vec{v}_i + \sum_{i=k+1}^{n} b_i \vec{v}_i = \vec{0}$

$\because$ $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for V and so it is lin. ind.

$\therefore$ $(-c_i) = 0$ for $i = 1, 2, \dots, k$

$b_i = 0$ for $i = k+1, k+2, \dots, n$

$\therefore$ S is lin. ind.

$\therefore$ S is basis for $R(T)$.

$$\therefore \quad \text{Nullity}(T) + \text{Rank}(T)$$

$$= k + (n-k)$$

$$= n = \dim(V)$$